Asymptotic expansion for the models of nonlinear dispersive, dissipative equations

Raúl Prado
Departamento de Matemática Universidade Federal de Paraná,
Caixa Postal 19081, CEP 81531-990 Curitiba, PR, Brazil
rprado@mat.ufpr.br

February 1, 2008

Abstract

Considered herein are the family of nonlinear equations with both dispersive and dissipative homogeneous terms appended. Solutions of these equations that start with finite energia decay to zero as time goes to infinity. We present an asymptotic form which renders explicit the influence of the dissipative, dispersive and nonlinear effect in this decay. We obtain the second term in the asymptotic expansion, as time goes to infinity, of the solutions of this equations and the complete asymptotic expansion, as time goes to infinity, of the linearized equations.

1 Introduction

This paper is intended to study the asymptotic expansion of solutions of one family of nonlinear, dispersive equations under the effect of dissipation. Our model equation takes the form

$$\begin{cases} u_t + Mu_t + Mu + u_x + \left(u^q\right)_x = 0, \quad x \in I\!R, \quad t > 0; \\ \quad u(x,0) = u_0(x), \end{cases} \tag{1.1}$$

where M is defined as Fourier multiplier homogeneus operator by

$$\widehat{Mf}(\xi) = |\xi|^m \widehat{f}(\xi), \quad f \in H^m(I\!R) \quad m \geq 1, \tag{1.2}$$

The circumflexes connote Fourier transform, and subscripts denote partial differentiation. When equations of the class (1.1) arise as models of phisical phenomena $u = u(x,t)$ represents the displacement of the medium of propagation from its equilibrium position and is a real-valued function of two real variables: x (called the spatial variable) is proportional to distance in the direction of propagation and $t > 0$ is proportional to time.

In this paper, u^q schould be interpreted either as $|u|^q$ or $|u|^{q-1}\ u$. We shall assume that $q > m > 2$. All physical constant which may appear in (1.1) are put to be equal to 1, for simplicity.

Equations of the form (1.1) arise when dissipation, dispersion, and the effect of nonlinearity are appended to the *transport equation* $u_t + u_x$ for the unidirectional wave propagation. The damping is represented here by Mu. When $Mu = 0$, this models arise in a wide variety of circunstances (see Biler [7], Benjamin [3, 4, 5] , Bona [10, 11] Abdelouhab et al [1]). The particular class in which the nonlinearity is a monomial, the dispersive and dissipative terms are homogeneus, provides perhaps the simplest class of model in which to study the three effects. The equation (1.1) is a more simple model of a general case than were studied by V. Bisognin and G. Perla in [9], decay rates of the solutions in $I\!L^p(I\!R)$ spaces, $2 \leq p \leq \infty$, were obtained.

In the case $m = 2$, i.e. when $Mu = -u_{xx}$, then the equation (1.1) is the well-known generalized Benjamin Bona Mahony Burger equation:

$$u_t - u_{xxt} - u_{xx} + u_x + (u^q)_x = 0, \quad x \in I\!R, \quad t > 0,$$

this model appear when one attempt to describe the propagation of small-amplitude long waves in nonlinear dispersive media taking into account dissipative mechanisms. For solutions of this equations G. Karch obtain in [23] the first an second terms of the asymptotic expansion, when $t \to \infty$, of both linearized equation and nonlinear equation.

In [27], we obtain the complete asymptotic expansion of solutions the linearized equation (the n-dimensional case), and compute the second term in the asymptotic expansion in the two-dimensional case, with quadratic nonlinear term.

When $m = 4$ the equation (1.1) takes the form

$$u_t + u_{xxxxt} + u_{xxxx} + u_x + (u^q)_x = 0, \quad x \in I\!R, \quad t > 0,$$

this is the Roseneau equation, with the term u_{xxxx} associated with dissipative phenomena. The Cauchy problem for this equation were solved by M. Park in [26]. The asymptotic expansion of solutions, when $t \to \infty$, is a consequence of our results.

For the well-posedness of the initial-value problem (1.1), we refer to Bisognin and Perla [9]. It is sufficient to know for the purpose of this paper that it is always possible to construct global in time solutions for any initial data $u_0 \in W^{2,1}(I\!R)$ provided either $\|u_0\|_{W^{2,1}(I\!R)}$ is small or some restriction on q are imposed. In the Section 6, we put this problem more carefully.

This paper is organized as follows. In the next Section we state and discuss main results concerning to the equation (1.1). In Section 3 we prove preliminary result of the complete asymptotic expansion of solutions to the equation of type (1.1) linearized, we will use essentially the Taylor Theorem, the Plancherel equality, and the complete asymptotic expansion for heat equation of generalized type. In section 4, we prove the complete asymptotic expansion of the linearized equation (1.1) using essentially, some properties of the Bessel Potential of order β, K_β. Section 5 contains a result on the complete asymptotic expansion of generalized type KdV-B

linear equation where the same techniques apply. For completeness of the exposition, the global-in-time solutions to (1.1) are constructed in Section 6. In Section 7 we calculate the second term in the asymptotic expansion, when $t \to \infty$, of the solution to the nonlinear equation (1.1). The proof bases on the same ideas as those in G. Karch [23].

Notation. The notation to be used is standard. For $1 \le p \le \infty$, the $I\!\!L^p(I\!\!R)$-norm of a Lebesgue measurable real-valued functions defined on $I\!\!R^n$ is denoted by $\|f\|_p$. The Fourier transform of u is given by $\mathcal{F}u(\xi) = \widehat{u}(\xi) \equiv \int_{I\!\!R} e^{-ix\cdot\xi}u(x)dx$. If $m \in I\!\!R$ we denote by $H^m(I\!\!R)$ the Sobolev space of order m as the completion of the Schwartz space $\mathcal{S}(I\!\!R)$ respect to the norm $\|u\|_{H^m} \equiv \left(\int_{I\!\!R}(1+|\xi|^2)^m|\widehat{u}(\xi)|^2 d\xi\right)^{1/2}$. For simplicity, we write $\int = \int_{I\!\!R}$. The letter C will denote generic positive constants, which do not depend on u, x and t, but may vary from line to line during computations.

2 Main result

In order to eliminate the convective term of order one in the equation (1.1), we define the traslated function $v(x,t) = u(x-t,t)$. Then u solves (1.1) if and only if v solves

$$\text{(2.3)} \qquad \begin{cases} v_t + Mv_t + Mv + Mv_x + (v^q)_x &= 0\,, \qquad in \quad I\!\!R \times (0,\infty)\,; \\ v(x,0) &= u_0(x) \qquad in \quad I\!\!R. \end{cases}$$

We now study the asymptotic development of the solution of equation (2.3).

The first goal of this paper is to analize the linear equation

$$\text{(2.4)} \qquad \begin{cases} v_t + Mv_t + Mv + Mv_x = 0, \quad x \in I\!\!R, \quad t > 0; \\ \quad v(x,0) = u_0(x), \end{cases}$$

and to obtain the asymptotic expansion of solutions, complete when m is an integer and until the second term when m is not integer respectively.

For the heat equation $u_t - \Delta u = 0$ in $I\!\!R^n$, this was done by J. Duoandixoetxea and E. Zuazua in [17]. Indeed, it was shown that if $u(x,t) = (G(t) * u_0)(x)$ is the solution of the heat equation whit initial data $u_0 \in I\!\!L^1(I\!\!R^n)$ and $G(x,t) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}}$ (G is called the heat kernel), with $u_0 \in I\!\!L^1(1+|x|^k)$ such that $|x|^{k+1}u_0(x) \in I\!\!L^p(I\!\!R^n)$, $1 \le p \le q \le \infty$, then

$$\text{(2.5)} \qquad \left\| G(t) * u_0 - \sum_{|\alpha| \le k} \frac{(-1)^{|\alpha|}}{\alpha!} \left(\int x^\alpha u_0(x)dx\right) D^\alpha G(t) \right\|_q \le Ct^{-(\frac{k+1}{2}) - \frac{n}{2}(\frac{1}{p}-\frac{1}{q})} \left\| \, |x|^{k+1}u_0 \right\|_p,$$

for all $t > 0$.

This shows that, for the solutions of the heat equation, a complete asymptotic expansion may be obtained by means of the moments of the initial data and using the derivatives of the Gaussian heat kernel as reference profiles.

In this work we show that for the solution of the linearized equation (2.4), besides the terms

$$\sum_{\alpha=0}^{k} \frac{(-1)^\alpha}{\alpha!} \left(\int x^\alpha v_0(x)dx\right) \partial_x^\alpha G_m(t)\,,$$

which correspond to the asymptotic expansion of the generalized linear heat equation

$$v_t - Mv = 0,$$

whit initial data $v_0 \in I\!\!L^1(I\!\!R)$ and where $G_m(x,t) = (1/(2\pi)) \int_{I\!\!R} e^{ix.\xi - t|\xi|^m} d\xi$, other terms due to the dispersive effects appear in its asymptotic expansion.

We shall denote by $S(t)v_0$ the solution to the equation (2.4), the function K_m is defined through its Fourier transform $\widehat{K_m}(\xi) = 1/(1+|\xi|^m)$, and $K_m^j = \underbrace{K_m * K_m * ... * K_m}_{\text{j}-\text{veces}}$.

Denoted

$$\mathcal{M}_\alpha(v_0) = \frac{(-1)^{|\alpha|}}{\alpha!} \left(\int x^\alpha v_0(x) dx \right),$$

The following Theorems holds:

Theorem 2.1 *Let $N \in I\!\!N$ and $m \in \mathbb{Z}^+$. Then there exist a constant $C = C(N) > 0$ such that*

$$\begin{aligned}
&\left\| S(t)v_0 - \sum_{\alpha=0}^{N} \mathcal{M}_\alpha(v_0) \partial_x^\alpha G_m(t) - \sum_{r=0}^{N} \frac{t^r}{r!} \sum_{j=0}^{[\frac{N}{2}]} \frac{t^j}{j!} \sum_{\substack{0 \le \alpha \le N-r-mj \\ (r,j)\neq(0,0)}}^{*} \mathcal{M}_\alpha(v_0) \big(\partial_x M\big)^r M^{2j} \partial_x^\alpha G_m(t) \right\|_2 \\
&\le C \Bigg[t^{-([\frac{N}{2}]+1)-\frac{1}{2m}} \|v_0\|_1 \left(\sum_{r=0}^{N} \frac{t^{-\frac{r}{m}}}{r!} \right) + t^{-\frac{N+1}{m}-\frac{1}{2m}} \|v_0\|_1 + t^{-\frac{N+1}{m}-\frac{1}{2m}} \sum_{r=1}^{N+1} \big\| |x|^r v_0 \big\|_1 + e^{-\frac{t}{2}} \|v_0\|_2 \\
&\qquad + e^{-\frac{t}{2}} t^{-\frac{1}{2m}} \left(\sum_{r=0}^{N} \frac{t^{-\frac{r}{m}}}{r!} \right) \left(\sum_{j=0}^{[\frac{N}{2}]} \frac{t^{-j}}{j!} \right) \|v_0\|_1 + \max_{0\le\alpha\le N-1} \Big\{ \big|\mathcal{M}_\alpha(v_0)\big| \Big\} t^{-\frac{1+N}{m}-\frac{1}{2m}} \big\| |x| K_m \big\|_1 \Bigg].
\end{aligned}$$

for all $v_0 \in I\!\!L^2(I\!\!R) \cap I\!\!L^1(I\!\!R; 1+|x|^{N+1})$.

When m is not integer, we still have the asymptotic expansion until the second term of $S(t)v_0$, indeed:

Theorem 2.2 *Let $N \in I\!\!N$ and $m = n + \delta$ for $n \in \mathbb{Z}^+$,$n > 1$ $0 < \delta < 1$. Then there exist a constant $C = C(N) > 0$ such that*

$$\begin{aligned}
&\left\| S(t)v_0 - \sum_{\alpha=0}^{1} \mathcal{M}_\alpha(v_0) \partial_x^\alpha G_m(t) - t\mathcal{M}_0(v_0) \big(\partial_x M\big) G_m(t) \right\|_2 \\
&\quad \le C \Bigg[t^{-\frac{2}{m}-\frac{1}{2m}} \|v_0\|_1 + t^{-\frac{2}{m}-\frac{1}{2m}} \sum_{r=1}^{2} \big\| |x|^r v_0 \big\|_1 + t^{-1-\frac{1}{2m}} \left(\sum_{r=0}^{1} \frac{t^{-\frac{r}{m}}}{r!} \right) \|v_0\|_1 + e^{-\frac{t}{2}} \|v_0\|_2 \\
&\qquad + e^{-\frac{t}{2}} t^{-\frac{1}{2m}} \left(\sum_{r=0}^{1} \frac{t^{-\frac{r}{m}}}{r!} \right) \|v_0\|_1 + \mathcal{M}_0(v_0)\, t^{-\frac{2}{m}-\frac{1}{2m}} \big\| |x| K_m \big\|_1 \Bigg]
\end{aligned}$$

for all $v_0 \in I\!\!L^2(I\!\!R) \cap I\!\!L^1(I\!\!R; 1+|x|^2)$.

Remark 2.3

(i) In the Theorem 2.1, for a real number $s>0$, $[s]=\max\{m\in I\!N\ ; m\leq s\}$ denotes its integer part and $\sum\limits_{|\alpha|\leq N-m-2j}^{*}$ means simply that the couples (r,j) such that $N-r-mj<0$ are not being considered in the sum.

(ii) Our result generalizes that obtained in [22] and [27] where the case $m=2$ in the equation (2.4) is studied.

(iii) In the Theorems 2.1, when m is a integer, we see two different terms in the expansion of $S(t)v_0$. The first one, $\sum\limits_{\alpha=0}^{N}\mathcal{M}_\alpha(v_0)\partial_x^\alpha G_m(t)$ appears also in the asymptotic expansion of the generalized heat equation. The second one is due to the dispersive phenomena.

(iv) In the Theorems 2.2, when m is not a integer, we see two different terms in the expansion of $S(t)v_0$. The first one, $\sum\limits_{\alpha=0}^{1}\mathcal{M}_\alpha(v_0)\partial_x^\alpha G_m(t)$ corresponding to the two first term in the asymptotic expansion of the generalized heat equation. The second one is $t\mathcal{M}_0(v_0)\big(\partial_x M\big)G_m(t)$ due to the dispersive phenomena caused by the operator $\partial_x M$.

We now study the nonlinear problem. That is, we consider the problem (2.3), whit the following basic assumption: $q>m$, $m>2$,and that the solutions of (2.3) satisfying the following decay estimates

$$\|v(t)\|_2\leq C(1+t)^{-\frac{1}{2m}}\ ,\quad \|v_x(t)\|_2\leq C(1+t)^{-\frac{1}{2m}-\frac{1}{m}}\ ,\quad \|v(t)\|_\infty\leq C(1+t)^{-\frac{1}{m}}, \tag{2.6}$$

for all $t>0$, the numbers C are independent of t.
We compute the second term of the asymptotic expansion of their solutions of the problem (2.3) when $t\to\infty$.
When $m=2$, in [22] it was shown the influence of the nonlinear term in the asymptotic expansion of their solutions. A result analogue when $m>2$ is the following Theorem:

Theorem 2.4 *Let $p\in[1,\infty)$, $q>m$, and $m>2$, and denote $\mathcal{M}=\int v_0(x)dx$. Suppose that u is a solution to (2.3), satisfying the decay estimates (2.6), with $v_0\in I\!L^1(I\!R)\cap H^2(I\!R)$, then*

i) *For* $m<q<m+1$,

$$t^{(q-\frac{1}{p})\frac{1}{m}+\frac{1}{m}}\left\|v(t)-S(t)v_0+\int_0^t\partial_x G_m(t-\tau)*(\mathcal{M}G_m(\tau))^q d\tau\right\|_p\to 0\ ,\quad when\quad t\to\infty\ .$$

ii) *For* $q=m+1$,

$$\frac{t^{(1-\frac{1}{p})\frac{1}{m}+\frac{1}{m}}}{\log t}\left\|v(t)-S(t)v_0+\log t\left(\int\Big(\mathcal{M}G_m(t)\Big)^{m+1}(x,1)dx\right)\partial_x G_m(t)\right\|_p\to 0,\quad when\ t\to\infty.$$

iii) *For* $q>m+1$,

$$t^{(1-\frac{1}{p})\frac{1}{m}+\frac{1}{m}}\left\|v(t)-S(t)v_0+\left(\int_0^\infty\int_{I\!R}v^q(y,\tau)dyd\tau\right)\partial_x G_m(t)\right\|_p\to 0\ ,\quad when\quad t\to\infty\ .$$

Of course, this result is to be complemented with Theorem 2.1 and 2.2 that providesa complete expansion of the linear component of $S(t)v_0$ to obtain a complete description of the first and second terms of v.
At this respect, we recall that, when $n = 1$ and $m = 2$, G. Karch in [23] studied the equation (2.3) and obtained the second order term for the cases of $2 < q < 3$, $q = 3$, $q > 3$. In this case $\int \bigl(\mathcal{M}G_2(t)\bigr)^3(x,1)dx = \mathcal{M}^3/(4\pi\sqrt{3})$.

Remark 2.5 *The condition (2.6) is not particularly restrictive, and is imposed for brevity sake. Indeed, see for example the work [9]. Moreover, for the completeness of our exposition, we construct solutions to (2.3) satisfying (2.6) provides v_0 is small in $W^{2,1}(I\!\!R)$ (see section 6).*

3 Linearized equation

In this section, we consider the family linearized (2.4) of dispersive equations under the effect of dissipation.
Using the Fourier transform we obtain that each solution to (2.4) with $u_0 \in \mathcal{S}'$ has the form

$$v(x,t) = S(t)v_0(x) = \frac{1}{2\pi}\int_{I\!\!R} \mathrm{e}^{-t\Phi\xi + ix.\xi}\widehat{v_0}\xi d\xi \tag{3.7}$$

whit the phase function

$$\Phi(\xi) = \frac{|\xi|^m - i\xi|\xi|^m}{1+|\xi|^m}$$

Our main goal will be to find the complete asymptotic expansion of solutions $S(t)v_0(x)$ as t approaches infinity, with this aim, first we obtain in the following subsection, the complete asymptotic expansion of solutions, as time goes to infinity, of the generalized heat equation.

3.1 Generalized heat equation. Complete asymptotic expansion.

The complete asymptotic expansion for the solution of heat equation were obtained by J. Duoandikoetxea and E. Zuazua in [17], in similar form, we have the complete asymptotic expansion for a generalization of the heat equation, our generalization takes the form

$$\begin{cases} u_t + Mu = 0, \quad x \in I\!\!R, \quad t > 0; \\ \quad u(x,0) = u_0(x), \end{cases} \tag{3.8}$$

Where the operator M is given by (1.2), with $m \geq 1$.
If $u_0 \in I\!\!L^1(I\!\!R)$, using the Fourier transform in the variable x, we obtain that the solution to (3.8) is a convolution product: $u(x,t) = G_m(t) * u_0(x)$, where

$$G_m(x,t) = \frac{1}{2\pi}\int_{I\!\!R} e^{-|\xi|^m t + ix\cdot\xi} d\xi, \tag{3.9}$$

is the heat kernel generalized.

If $t > 0$ and j is a nonnegative integer, then using the Euler integral it can be easily seen that

$$\int_{I\!R} \xi^{2j} e^{-2t\xi^m} d\xi = C(m,j) t^{-\frac{2j}{m}-\frac{1}{m}}, \quad \text{where} \quad C(m,j) = \frac{\Gamma(\frac{2j+1}{m})}{m}.$$

Hence, it follows that

$$\left\|\partial_x^j G_m(t)\right\|_2 = C(m,j) t^{-\frac{j}{m}-\frac{1}{2m}} \tag{3.10}$$

In general we have the following result:

$$\left\|\partial_x^j G_m(t)\right\|_p \leq C t^{-\frac{j}{m}-\frac{1}{m}(1-\frac{1}{p})}, \quad \forall\, 1 \leq p \leq \infty, \quad m \geq 1. \tag{3.11}$$

For the proof of (3.11) we estimate $\left\|\partial^j G_m(t)\right\|_\infty$ and $\left\|\partial^j G_m(t)\right\|_1$, then interpolation. The case $p = 1$, use the following formula (ver [21])

$$\left\|w\right\|_1 \leq C \left\|w\right\|_2^{1/2} \left\|\partial_x \widehat{w}\right\|_2^{1/2}.$$

We have the complete asymptotic expansion for the solutions of (3.8) by means of the moments of the initial data and using the derivatives of the generalized heat kernel as reference profiles. Indeed, if denote the moments of the initial data by

$$\mathcal{M}_j(u_0) = \frac{(-1)^j}{j!} \int_{I\!R} x^j u_0(x) dx,$$

we have the following Theorem:

Theorem 3.1 *Assume that u is the solution to (3.8), with $u_0 \in I\!L^1(I\!R)$. Let j, N nonnegatives integer. Then, there exist a constant $C > 0$ such that*

$$\left\|G_m(t) * u_0 - \sum_{j=0}^{N} \mathcal{M}_j(u_0) \partial_x^j G_m(t)\right\|_p \leq C t^{-\frac{1}{m}(1-\frac{1}{p})-\frac{N+1}{m}} \left\||x|^{N+1} u_0\right\|_1, \quad \forall p \in [1,\infty], \tag{3.12}$$

for all $u_0 \in I\!L((1+|x|^N)dx, I\!R) \cap I\!L^1(|x|^{N+1}dx, I\!R)$

Proof.-By [17] u_0 can be decomposed as

$$u_0 = \sum_{j=0}^{N} \mathcal{M}_j(u_0) \partial_x^j \delta + \partial_x^{N+1} F_{N+1},$$

where $F_{N+1} \in I\!L^1(I\!R)$, such that $\left\|F_{N+1}\right\|_1 \leq \left\||x|^{N+1} u_0\right\|_1$.

Hence, taking the convolution of u_0 with $G_m(t)$, we obtained

$$\begin{aligned}\left\|G_m(t) * u_0 - \sum_{j=0}^{N} \mathcal{M}_j(u_0) \partial_x^j G_m(t)\right\|_p = \left\|\partial^{N+1} F_{N+1} * G_m(t)\right\|_p &\leq \left\|F_{N+1}\right\|_1 \left\|\partial^{N+1} G_m(t)\right\|_p \\ &\leq C t^{-\frac{N+1}{m}-\frac{1}{m}(1-\frac{1}{p})} \left\||x|^{N+1} u_0\right\|_1. \qquad \square\end{aligned}$$

The following result gives the different representations of the operator M

Lemma 3.2 *([16]) Let M the operator defined by $\widehat{Mu}(\xi) = |\xi|^m \widehat{u}(\xi)$, for $u \in H^r$, where $0 < m \leq r$, r is a positive integer. It follows that*

1. *let $m = 2n$ for $n \in Z^+$, then*

$$Mu(x) = (-1)^n \partial_x^{2n} u(x);$$

2. *let $m = 2n+1$ for $n \in Z^+$, then*

$$Mu(x) = (-1)^n \sqrt{\frac{2}{\pi}} \int \frac{\partial_y^{2n+1} u(y)}{x-y} dy;$$

3. *let $m = 2n+\delta$ for $n \in Z^+$, $0 < \delta < 1$, then*

$$Mu(x) = (-1)^n \sqrt{\frac{\pi}{2}} \Big(cos(\frac{\delta\pi}{2}) \Gamma(1-\delta) \Big)^{-1} \int sign(x-y) \frac{\partial_y^{2n+1} u(y)}{|x-y|^\delta} dy,$$

   where Γ denote the gamma function;

4. *let $m = 2n+1+\delta$ for $n \in Z^+$, $0 < \delta < 1$, then*

$$Mu(x) = (-1)^n \sqrt{\frac{\pi}{2}} \Big(sen(\frac{\delta\pi}{2}) \Gamma(1-\delta) \Big)^{-1} \int \frac{\partial_y^{2n+1} u(y)}{|x-y|^\delta} dy.$$

3.2 Complete asymptotic expansion: linear case

In this section we obtain preliminary result on the asymptotic development of solutions $S(t)v_0$ of (2.4). In this developmente, appear terms of convolution $K_m^{r+j} * (\partial_x M)^r M^{2j} \partial_x^\alpha G_m(t)$, then in the following section, to prove Theorems 2.1 and 2.2 we replace them by the first term of their asymptotic expansion, i.e. by $(\partial_x M)^r M^{2j} \partial_x^\alpha G_m(t)$.
The following preliminary Theorems holds:

Theorem 3.3 *Let $N \in \mathbb{N}$ and $m \in \mathbb{Z}^+$. Then, there exist a constant $C = C(N) > 0$ such that*

$$\begin{aligned}
&\Big\| S(t)v_0 - \sum_{r=0}^{N} \frac{t^r}{r!} \sum_{j=0}^{[\frac{N}{2}]} \frac{t^j}{j!} \sum_{0 \leq \alpha \leq N-r-mj}^{*} \mathcal{M}_\alpha(v_0) K_m^{r+j} * (\partial_x M)^r M^{2j} \partial_x^\alpha G_m(t) \Big\|_2 \\
&\leq C t^{-(N+1)\frac{1}{m} - \frac{1}{2m}} \big\| v_0 \big\|_1 + C t^{-([\frac{N}{2}]+1) - \frac{1}{2m}} \Big(\sum_{r=0}^{N} \frac{t^{-\frac{r}{m}}}{r!} \Big) \big\| v_0 \big\|_1 \\
&+ C t^{-(N+1)\frac{1}{m} - \frac{1}{2m}} \sum_{r=0}^{N} \sum_{\substack{0 \leq j \leq [\frac{N}{2}] \\ 0 \leq j \leq [\frac{N-r}{m}]}} \Big\| |x|^{N+1-r-jm} v_0 \Big\|_1 \\
&+ C e^{-\frac{t}{2}} \big\| v_0 \big\|_2 + C e^{-\frac{t}{2}} t^{-\frac{1}{2m}} \Big(\sum_{r=0}^{N} \frac{t^{-\frac{r}{m}}}{r!} \Big) \Big(\sum_{j=0}^{[\frac{N}{2}]} \frac{t^{-j}}{j!} \Big) \big\| v_0 \big\|_1
\end{aligned}$$

for all $v_0 \in \mathbb{L}^2(\mathbb{R}) \cap \mathbb{L}^1(\mathbb{R}^n; 1 + |x|^{N+1})$.

When m is not integer, we still have the asymptotic expansion until the second term of $S(t)v_0$ indeed, we have:

Theorem 3.4 *Let $N \in I\!N$ and $m = n + \delta$, for $n \in \mathbb{Z}^+$, $0 < \delta < 1$. Then, there exis a constant $C = C(N) > 0$ such that*

$$\left\| S(t)v_0 - \sum_{\alpha=0}^{1} \mathcal{M}_\alpha(v_0)\partial_x^\alpha G_m(t) - tMK_m * \partial_x M \right\|_2 \le Ct^{-\frac{2}{m}-\frac{1}{2m}} \|v_0\|_1$$
$$+ Ct^{-\frac{2}{m}-\frac{1}{2m}} \sum_{r=1}^{2} \||x|^r v_0\|_1 + Ct^{-1-\frac{1}{2m}} \left(\sum_{r=0}^{1} \frac{t^{-\frac{r}{m}}}{r!} \right) \|v_0\|_1$$
$$+ Ce^{-\frac{t}{2}} \|v_0\|_2 + Ce^{-\frac{t}{2}} t^{-\frac{1}{2m}} \left(\sum_{r=0}^{1} \frac{t^{-\frac{r}{m}}}{r!} \right) \|v_0\|_1,$$

for all $v_0 \in I\!L^2(I\!R) \cap I\!L^1(I\!R^n; 1 + |x|^2)$.

For the proof of Theorems 3.3 and 3.4, we will use the following Lemmas. We consider $\varphi \in \mathcal{C}_0^\infty(I\!R)$, a cut-off function such that:

$$\begin{aligned} |\varphi(\xi)| &\le 1, \quad \forall \xi \in I\!R^n ; \\ \varphi(\xi) &= 1, \quad \text{if} \quad |\xi| \le 1 ; \\ \varphi(\xi) &= 0, \quad \text{if} \quad |\xi| > 2. \end{aligned}$$

Then, we define

$$S_\varphi(x,t) = \frac{1}{2\pi} \int e^{-t\Phi_0(\xi) + ix\cdot\xi} \varphi(\xi) d\xi. \tag{3.13}$$

Where

$$\Phi_0(\xi) = \frac{|\xi|^m}{1 + |\xi|^m}, \quad m \ge 1.$$

Notice that if define

$$K_m^j = \underbrace{K_m * K_m * ... * K_m}_{\text{j−veces}},$$

where $\widehat{K}_m(\xi) = 1/(1 + |\xi|^m)$, then

$$\left[\widehat{K}_m(\xi)\right]^j = \widehat{K_m^j}(\xi) = \frac{1}{\left[1 + |\xi|^m\right]^j}. \tag{3.14}$$

Hence

$$\begin{aligned} \frac{1}{2\pi} \int e^{-t\Phi_0(\xi) + ix\cdot\xi} \, \widehat{u}_0(\xi) d\xi &= \frac{1}{2\pi} \int e^{-t|\xi|^m + ix\cdot\xi} \, e^{\frac{t|\xi|^{2m}}{1+|\xi|^m}} \, \widehat{u}_0(\xi) d\xi \\ &= \sum_{j=0}^{\infty} \frac{1}{2\pi} \int e^{-t|\xi|^m + ix\cdot\xi} \left(t|\xi|^{2m} \, \widehat{K_m \xi} \right)^j \frac{1}{j!} \, \widehat{u}_0(\xi) d\xi \\ &= \sum_{j=0}^{\infty} \frac{t^j}{j!} \left(K_m^j * M^{2j} G_m * u_0 \right)(x). \end{aligned}$$

Then, we have the following results:

Lemma 3.5 *Denote* $K_m^j = \underbrace{K_m * K_m * ... * K_m}_{\text{j−veces}}$, $m \geq 1$. *Let* $N \in I\!N$. *There exist a constant* $C = C(N,m)$ *such that*

$$\left\| S_\varphi(t) * u_0 - \sum_{j=0}^{N} \frac{t^j}{j!} K_m^j * M^{2j} G_m * u_0 \right\|_2 \leq C t^{-(N+1)-\frac{1}{2m}} \|u_0\|_1 + C e^{-t/2} t^{-\frac{1}{2m}} \left(\sum_{j=0}^{N} \frac{t^{-j}}{j!} \right) \|u_0\|_1 , \tag{3.15}$$

for all $t > 0$ *and* $u_0 \in I\!L^1(I\!R)$.

Proof.-We decompose $G_m(x,t)$ using the cut-off function $\varphi(\xi)$

$$G_m(x,t) = G_{m\varphi}(x,t) + G_{m(1-\varphi)}(x,t).$$

Then

$$\left\| S_\varphi(t) * u_0 - \sum_{j=0}^{N} \frac{t^j}{j!} K_m^j * M^{2j} G_m * u_0 \right\|_2 \leq \left\| \sum_{j=0}^{N} \frac{t^j}{j!} K_m^j * M^{2j} G_{m(1-\varphi)} * u_0) \right\|_2 + \left\| S_\varphi(t) * u_0 - \sum_{j=0}^{N} \frac{t^j}{j!} K_m^j * M^{2j} G_{m\varphi} * u_0 \right\|_2 = I_1(t) + I_2(t). \tag{3.16}$$

We estimate $I_i(t)$, $i = 1, 2$, separately.
The term $I_2(t)$: Let

$$w(x,t) = S_\phi(t) - \sum_{j=0}^{N} \frac{t^j}{j!} K_m^j * M^{2j} G_{m\varphi}.$$

Then, its Fourier transform is

$$\begin{aligned} \widehat{w}(\xi,t) &= e^{-\frac{t|\xi|^m}{1+|\xi|^m}} \varphi(\xi) - \sum_{j=0}^{N} \frac{t^j}{j!} \left(\frac{|\xi|^{2m}}{1+|\xi|^m} \right)^j e^{-t|\xi|^m} \varphi(\xi) \\ &= e^{-t|\xi|^m} \varphi(\xi) \left[e^{\frac{t|\xi|^{2m}}{1+|\xi|^m}} - \sum_{j=0}^{N} \frac{t^j}{j!} \left(\frac{|\xi|^{2m}}{1+|\xi|^m} \right)^j \right] . \end{aligned} \tag{3.17}$$

Hence, using the Taylor expansion of the exponential function, we have

$$e^x - \sum_{j=0}^{N} \frac{x^j}{j!} \leq \frac{x^{N+1}}{(N+1)!} e^x.$$

Using this inequality in (3.17) we obtain

$$\begin{aligned} \left| \widehat{w}(\xi,t) \right| &\leq e^{-t|\xi|^m} \varphi(\xi) \left(t \frac{|\xi|^{2m}}{1+|\xi|^m} \right)^{N+1} e^{\frac{t|\xi|^{2m}}{1+|\xi|^m}} \\ &\leq C_N \, \varphi(\xi) \, e^{-\frac{t|\xi|^m}{1+|\xi|^m}} \, \left(t \frac{|\xi|^{2m}}{1+|\xi|^m} \right)^{N+1} . \end{aligned} \tag{3.18}$$

By Plancherel's formula we see that

$$
(3.19) \qquad \left\|w(t)\right\|_2^2 = \int \left|\widehat{w}\right|^2 d\xi \leq C_N \int |\varphi(\xi)|^2 e^{-\frac{2t|\xi|^m}{1+|\xi|^m}} \left(\frac{t|\xi|^{2m}}{1+|\xi|^m}\right)^{2(N+1)} d\xi
$$

$$
\leq Ct^{2(N+1)} \int_{|\xi|\leq 1} e^{-\frac{2t|\xi|^m}{1+|\xi|^m}} \frac{|\xi|^{4m(N+1)}}{1+|\xi|^m} d\xi
$$

$$
\leq Ct^{2(N+1)} \int e^{-t|\xi|^m} \left|\xi\right|^{4m(N+1)} d\xi, \ (m \geq 1, \ |\xi| \leq 1 \Rightarrow \frac{2|\xi|^m}{1+|\xi|^m} \geq |\xi|^m)
$$

$$
\leq Ct^{2(N+1)} t^{-\frac{4m(N+1)}{m} - \frac{1}{m}} = Ct^{-2(N+1)-\frac{1}{m}}.
$$

Then, from (3.19) we have

$$
(3.20) \qquad I_2(t) \leq \left\| S_\varphi * u_0 - \sum_{j=0}^{N} \frac{t^j}{j!} K_m^j * M^{2j} G_{m\varphi} \right\|_2 \left\|u_0\right\|_1 \leq Ct^{(N+1)-\frac{1}{2m}} \left\|u_0\right\|_1.
$$

The term $I_1(t)$:
Given that $\left\|K_m^j\right\|_1 = 1$, then we have

$$
(3.21) \qquad I_1(t) \leq \sum_{j=0}^{N} \frac{t^j}{j!} \left\|K_m^j\right\|_1 \left\|M^{2j} G_{m(1-\varphi)}\right\|_2 \left\|u_0\right\|_1 \leq \sum_{j=0}^{N} \frac{t^j}{j!} \left\|M^{2j} G_{m(1-\varphi)}\right\|_2 \left\| u_0\right\|_1.
$$

Now, by Plancherel's formula and using that $|\xi| \geq 1$ implies $2|\xi|^m \geq 1 + |\xi|^m$, we have

$$
(3.22) \qquad \left\|M^{2j} G_{m(1-\varphi)}\right\|_2^2 = \int \left|1 - \varphi(\xi)\right|^2 \left|\xi\right|^{4jm} e^{-2t|\xi|^m} d\xi \leq \int_{|\xi|\geq 1/2} \left|\xi\right|^{4mj} e^{-2t|\xi|^m} d\xi
$$

$$
\leq e^{-t} \int e^{-t|\xi|^m} |\xi|^{4mj} d\xi \leq Ce^{-t} t^{-4j-\frac{1}{m}}.
$$

Then, using (3.22) in (3.21) we have

$$
(3.23) \qquad I_1(t) \leq Ce^{-t/2} t^{-\frac{1}{2m}} \sum_{j=0}^{N} \frac{t^{-j}}{j!} \left\|u_0\right\|_1.
$$

Then, from (3.23) and (3.20) we have (3.15). □

Lemma 3.6 *Let $N \in I\!N$. Then there exist a constant positive $C = C(N, m)$ such that*

$$
(3.24) \qquad \left\| S(t)u_0 - \sum_{j=0}^{N} \frac{t^j}{j!} K_m^j * \left(\partial_x M\right)^j S_\varphi * u_0 \right\|_2 \leq C\Big[t^{-\frac{N+1}{m} - \frac{1}{2m}} \left\|u_0\right\|_1 + e^{-t/2} \left\|u_0\right\|_2\Big],
$$

for all $t > 0$ and $u_0 \in I\!L^2(I\!R) \cap I\!L^1(I\!R)$

Proof.-We decompose (t) using the cut-off function φ, we have

$$
(3.25) \quad S(t)u_0 - \sum_{j=0}^{N} \frac{t^j}{j!} K_m^j * \left(\partial_x M\right)^j S_\varphi * u_0 = \frac{1}{2\pi} \int e^{t\widetilde{\Phi} + ix\cdot\xi} \, \widehat{u_0}(\xi) \big(1 - \varphi(\xi)\big) \, d\xi
$$

$$
+ \frac{1}{2\pi} \int e^{t\widetilde{\Phi} + ix\cdot\xi} \, \widehat{u_0}(\xi) \varphi(\xi) d\xi - \sum_{j=0}^{N} \frac{t^j}{j!} K_m^j * \left(\partial_x M\right)^j S_\varphi * u_0(x) = J_1(t) + J_2(t).
$$

Here $J_2(t)$ represents the difference of the last two terms in (3.25). We estimate $J_i(t)$, $i = 1, 2$, separately.

The term $J_2(t)$:

The Fourier transform of $J_2(x,t)$ is

$$\begin{aligned}\widehat{J_2}(\xi,t) &= e^{-t\widetilde{\Phi}}\varphi(\xi)\widehat{u_0}(\xi) - \sum_{j=0}^{N}\frac{t^j}{j!}\bigg(\frac{-i\xi|\xi|^m}{1+|\xi|^m}\bigg)^j e^{-\frac{t|\xi|^m}{1+|\xi|^m}}\varphi(\xi)\widehat{u_0}(\xi)\\ &= e^{-\frac{t|\xi|^m}{1+|\xi|^m}}\varphi(\xi)\widehat{u_0}(\xi)\Bigg[e^{-\frac{it\xi|\xi|^m}{1+|\xi|^m}} - \sum_{j=0}^{N}\frac{t^j}{j!}\bigg(-\frac{i\xi|\xi|^m}{1+|\xi|^m}\bigg)^j\Bigg].\end{aligned}$$

The Taylor expansion of e^{ix} implies that $\big|e^{ix} - \sum_{j=0}^{N}\frac{(ix)^j}{j!}\big| \le \frac{x^{N+1}}{(N+1)!}$. Then

$$\big|\widehat{J_2}(\xi,t)\big| \le Ce^{-\frac{t|\xi|^m}{1+|\xi|^m}}\,\big|\varphi(\xi)\big|\big|\widehat{u_0}(\xi)\big|\bigg(\frac{t\xi|\xi|^m}{1+|\xi|^m}\bigg)^{N+1} \tag{3.26}$$

By Plancherel's formula, by (3.26) and observing that $|\xi| \le 1$ implies $\frac{2|\xi|^m}{1+|\xi|^m} \ge |\xi|^m$, $(m \ge 1)$, we have

$$\begin{aligned}\big\|J_2(t)\big\|_2^2 &= \int\big|\widehat{J_2}(\xi)\big|^2 d\xi \le C_N\int e^{-\frac{2t|\xi|^m}{1+|\xi|^m}}\big|\varphi(\xi)\big|^2\big|\widehat{u_0}(\xi)\big|^2\big(t\xi|\xi|^m\big)^{2(N+1)}d\xi\\ &\le C_N t^{2(N+1)}\big\|u_0\big\|_1^2\int_{|\xi|\le 1} e^{-\frac{2t|\xi|^m}{1+|\xi|^m}}|\xi|^{2(m+1)(N+1)}\\ &\le Ct^{2(N+1)}\big\|u_0\big\|_1^2\int e^{-t|\xi|^m}\big|\xi\big|^{2(m+1)(N+1)}d\xi \le Ct^{-\frac{2(N+1)}{m}-\frac{1}{m}}\big\|u_0\big\|_1^2.\end{aligned} \tag{3.27}$$

The term $J_1(t)$:

The Fourier transforme of $J_1(x,t)$ is

$$\widehat{J_1}(\xi) = e^{-t\varphi(\xi)}\ \widehat{u_0}(\xi)\ \big(1-\varphi(\xi)\big).$$

Then, by Plancherel's formula we have

$$\begin{aligned}\big\|J_1(t)\big\|_2^2 &= \int\big|\widehat{J_1}(\xi,t)\big|^2 d\xi = \int e^{\frac{-2t|\xi|^m}{1+|\xi|^m}}\big|\widehat{u_0}(\xi)\big|^2\big|1-\varphi(\xi)\big|^2 d\xi\\ &\le \int_{|\xi|\ge 1/2} e^{\frac{-2t|\xi|^m}{1+|\xi|^m}}\big|\widehat{u_0}(\xi)\big|^2 d\xi.\end{aligned} \tag{3.28}$$

Now, we have that if $m \ge 1$, then

$$|\xi| \ge 1 \Rightarrow 2|\xi|^m \ge 1 + |\xi|^m$$

Returning to (3.28), we have

$$\big\|J_1(t)\big\|_2^2 \le \int e^{-t}\big|u_0(\xi)\big|^2 d\xi = e^{-t}\big\|u_0\big\|_2^2. \tag{3.29}$$

The Lemma 3.5 is consequence of (3.27) and (3.29). □

Lemma 3.7 *Let $r \in I\!N$, $r \geq 1$. Then there exist a constant $C = C(N,r) > 0$ such that*

$$\left\| K_m^r * \left(\partial_x M\right)^r S_\varphi(t) * v_0 - \sum_{j=0}^{N} \frac{t^j}{j!} K_m^{r+j} * \left(\partial_x M\right)^r M^{2j} G_m(t) * v_0 \right\|_2$$

$$\leq C t^{-\frac{1}{m}(r(m+1)+m(N+1))-\frac{1}{2m}} \left\|v_0\right\|_1 + C e^{-\frac{t}{2}} t^{-(\frac{m+1}{m})r-\frac{1}{2m}} \left(\sum_{j=0}^{N} \frac{t^{-j}}{j!} \right) \left\|v_0\right\|_1$$

for all $t > 0$ and $v_0 \in I\!L^1(I\!R)$.

Proof.-Given that $\left\|K_m^r\right\|_1 = 1$, we have

$$\left\| K_m^r * \left(\partial_x M\right)^r S_\varphi(t) * v_0 - \sum_{j=0}^{N} \frac{t^j}{j!} K_m^{r+j} * \left(\partial_x M\right)^r M^{2j} G_m(t) * v_0 \right\|_2 \leq$$

$$\leq \left\| \left(\partial_x M\right)^r S_\varphi(t) * v_0 - \sum_{j=0}^{N} \frac{t^j}{j!} K_m^{j} * \left(\partial_x M\right)^r M^{2j} G_m(t) * v_0 \right\|_2 .$$

Hence, we only need to prove that

$$\left\| \left(\partial_x M\right)^r S_\varphi(t) * v_0 - \sum_{j=0}^{N} \frac{t^j}{j!} K_m^{j} * \left(\partial_x M\right)^r M^{2j} G_m(t) * v_0 \right\|_2 \leq \tag{3.30}$$

$$\leq C t^{-\frac{1}{m}(r(m+1)+m(N+1))-\frac{1}{2m}} \left\|v_0\right\|_1 + C e^{-\frac{t}{2}} t^{-(\frac{m+1}{m})r-\frac{1}{2m}} \left(\sum_{j=0}^{N} \frac{t^{-j}}{j!} \right) \left\|v_0\right\|_1$$

We decompose G_m using the cut-off function φ:

$$G_m(x,t) = G_{m\varphi}(x,t) + G_{m(1-\varphi)}(x,t).$$

Then

$$\left\| \left(\partial_x M\right)^r S_\varphi(t) * v_0 - \sum_{j=0}^{N} \frac{t^j}{j!} K_m^{j} * \left(\partial_x M\right)^r M^{2j} G_m(t) * v_0 \right\|_2 \leq \tag{3.31}$$

$$\leq \left\| \left(\partial_x M\right)^r S_\varphi(t) * v_0 - \sum_{j=0}^{N} \frac{t^j}{j!} K_m^{j} * \left(\partial_x M\right)^r M^{2j} G_{m\varphi}(t) * v_0 \right\|_2$$

$$+ \left\| \sum_{j=0}^{N} \frac{t^j}{j!} K_m^{j} * \left(\partial_x M\right)^r M^{2j} G_{m(1-\varphi)}(t) * v_0 \right\|_2 = I_1(t) + I_2(t) .$$

We estimate $I_1(t)$ e $I_2(t)$ separately.

The term $I_1(t)$: We have

$$\left\| \left(\partial_x M\right)^r S_\varphi(t) * v_0 - \sum_{j=0}^{N} \frac{t^j}{j!} K_m^{j} * \left(\partial_x M\right)^r M^{2j} G_{m\varphi}(t) * v_0 \right\|_2 \leq \tag{3.32}$$

$$\leq \left\| \left(\partial_x M\right)^r S_\varphi(t) - \sum_{j=0}^{N} \frac{t^j}{j!} K_m^{j} * \left(\partial_x M\right)^r M^{2j} G_{m\varphi}(t) \right\|_2 \left\|v_0\right\|_1 .$$

We now get, $g(x,t) = \big(\partial_x M\big)^r S_\varphi(t) - \sum\limits_{j=0}^{N} \frac{t^j}{j!} K_m^j * \big(\partial_x M\big)^r M^{2j} G_{m\varphi}(t)$. Its Fourier transform is

$$\text{(3.33)} \qquad \begin{aligned} \widehat{g}(\xi,t) &= \big(i\xi|\xi|^m\big)^r \varphi(\xi) \left(e^{-\frac{t|\xi|^m}{1+|\xi|^m}} - \sum_{j=0}^{N} \frac{t^j}{j!} \Big(\widehat{K_m}(\xi)\, |\xi|^{2m} \Big)^j e^{-t|\xi|^m} \right) \\ &= \varphi(\xi) \big(i\xi|\xi|^m\big)^r e^{-t|\xi|^m} \left[e^{\frac{t|\xi|^{2m}}{1+|\xi|^m}} - \sum_{j=0}^{N} \frac{t^j}{j!} \left(\frac{|\xi|^{2m}}{1+|\xi|^m} \right)^j \right]. \end{aligned}$$

Now, using the Taylor expansion of the function $e^x, x \geq 0$, we have $e^x - \sum_{j=0}^N \frac{x^j}{j!} \leq \frac{x^{N+1}}{(N+1)!} e^x$. Using this inequality in (3.33) we have

$$\text{(3.34)} \qquad \begin{aligned} \big|\widehat{g}(\xi,t)\big| &\leq C\,\varphi(\xi)\, |\xi|^{r(m+1)} \frac{e^{-t|\xi|^m}}{(N+1)!} \left(\frac{t|\xi|^{2m}}{1+|\xi|^m} \right)^{N+1} e^{\frac{t|\xi|^{2m}}{1+|\xi|^m}} \\ &= C\,\varphi(\xi)\, |\xi|^{r(m+1)} \frac{e^{-\frac{t|\xi|^m}{1+|\xi|^m}}}{(N+1)!} \left(\frac{t|\xi|^{2m}}{1+|\xi|^m} \right)^{N+1}. \end{aligned}$$

¿From (3.34) and Plancherel's formula we obtain

$$\text{(3.35)} \qquad \begin{aligned} &\left\| \big(\partial_x M\big)^r S_\varphi(t) - \sum_{j=0}^{N} \frac{t^j}{j!} K_m^j * \big(\partial_x M\big)^r M^{2j} G_{m\varphi}(t) \right\|_2^2 = \int \big|\widehat{g}(\xi,t)\big|^2 d\xi \\ &\leq C_N \int \big|\varphi(\xi)\big|^2 |\xi|^{2r(m+1)} e^{-\frac{2t|\xi|^m}{1+|\xi|^m}} \left(\frac{t|\xi|^{2m}}{1+|\xi|^m} \right)^{2(N+1)} d\xi \\ &= C_N t^{2(N+1)} \int_{|\xi|\leq 1} |\xi|^{2r(m+1)+4m(N+1)} e^{-t|\xi|^m} d\xi \\ &\leq C_N t^{2(N+1)} \times t^{-\frac{2}{m}[r(m+1)+2m(N+1)]-\frac{1}{m}} = C t^{-\frac{2}{m}[r(m+1)+m(N+1)]-\frac{1}{m}}. \end{aligned}$$

In (3.35) we use that $|\xi| \leq 1$ implies $\frac{2|\xi|^m}{1+|\xi|^m} \geq |\xi|^m$. Then, returning (3.32), by (3.35) it follows that

$$\text{(3.36)} \qquad \begin{aligned} &\left\| \big(\partial_x M\big)^r S_\varphi(t) * v_0 - \sum_{j=1}^{N} \frac{t^j}{j!} K_m^j * \big(\partial_x M\big)^r M^{2j} G_{m\varphi}(t) * v_0 \right\|_2 \leq \\ &\qquad\qquad \leq C t^{-\frac{1}{m}[r(m+1)+m(N+1)]-\frac{1}{2m}} \big\|v_0\big\|_1 . \end{aligned}$$

The term $I_2(t)$: Since $\big\|K_m^j\big\|_1 = 1$, we have

$$\text{(3.37)} \qquad \left\| \sum_{j=0}^{N} \frac{t^j}{j!} K_m^j * \big(\partial_x M\big)^r M^{2j} G_{m(1-\varphi)} * v_0 \right\|_2 \leq \sum_{j=0}^{N} \frac{t^j}{j!} \big\| \big(\partial_x M\big)^r M^{2j} G_{m(1-\varphi)} \big\|_2 \big\|v_0\big\|_1 ,$$

and moreover if $m \geq 1$, $|\xi| \geq 1$ implies that $2t|\xi|^m \geq \big(t + t|\xi|^m\big)$. Then

$$\begin{aligned} \left\| \big(\partial_x M\big)^r M^{2j} G_{m(1-\varphi)}(t) \right\|_2^2 &= \int \Big(|\xi|^{r(m+1)+2mj} e^{-t|\xi|^m} \big(1-\varphi\big) \Big)^2 d\xi \\ &= \int_{|\xi|\geq 1} |\xi|^{2r(m+1)+4mj} e^{-2t|\xi|^m} d\xi \leq e^{-t} \int |\xi|^{2r(m+1)+4mj} e^{-t|\xi|^m} d\xi \\ &\leq C e^{-t} t^{-\frac{2}{m}[r(m+1)+2jm]-\frac{1}{m}} . \end{aligned}$$

Hence, returning to (3.37), we have

$$(3.38)\quad \Big\|\sum_{j=0}^{N}\frac{t^j}{j!}K_m^j*\big(\partial_x M\big)^r M^{2j}G_{m(1-\varphi)}(t)*v_0\Big\|_2 \le C\sum_{j=0}^{N}\frac{t^j}{j!}e^{-\frac{t}{2}}t^{-\frac{1}{m}[r(m+1)+2jm]-\frac{1}{2m}}\big\|v_0\big\|_1$$
$$\le Ce^{-\frac{t}{2}}t^{-\frac{r(m+1)}{m}-\frac{1}{2m}}\Big(\sum_{j=0}^{N}\frac{t^{-j}}{j!}\Big)\big\|v_0\big\|_1\,.$$

The Lemma 3.6 is consequence of (3.36) and (3.38). □

Lemma 3.8 *Let $(k,r,j)\in \mathbb{N}^3$. Then there exist a constant $C=C(k,r,j)>0$ such that*

$$(3.39)\quad \Big\|\big(\partial_x M\big)^r M^{2j}G_m(t)*v_0-\sum_{\alpha=0}^{k}\frac{(-1)^\alpha}{\alpha!}\Big(\int v_0x^\alpha dx\Big)\big(\partial_x M\big)^r M^{2j}\partial_x^\alpha G_m(t)\Big\|_2$$
$$\le Ct^{-\frac{1}{m}(r(m+1)+2mj+k+1)-\frac{1}{2m}}\big\||x|^{k+1}v_0\big\|_1\,,$$

for all $t>0$ and $v_0\in \mathbb{L}^1\big(\mathbb{R},1+|x|^{k+1}\big)$.

Proof.-In [17] it is proved that v_0 can be decomposed as

$$v_0=\sum_{\alpha=0}^{k}\frac{(-1)^\alpha}{\alpha!}\Big(\int v_0x^\alpha dx\Big)\partial_x^\alpha\delta+\partial_x^{k+1}F_{k+1}\,,$$

where $F_{k+1}\in \mathbb{L}^1(\mathbb{R})$, such that $\big\|F_{k+1}\big\|_1\le \big\||x|^{1+k}v_0\big\|_1$. Then, taking the convolution of v_0 with $\big(\partial_x M\big)^r M^{2j}G_m(x,t)$ we have

$$\Big\|\big(\partial_x M\big)^r M^{2j}G_m(t)*v_0-\sum_{\alpha=0}^{k}\frac{(-1)^\alpha}{\alpha!}\Big(\int v_0x^\alpha dx\Big)\big(\partial_x M\big)^r M^{2j}\partial_x^\alpha G_m(t)\Big\|_2$$
$$\le \big\|F_{k+1}*\big(\partial_x M\big)^r M^{2j}\partial_x^{k+1}G_m(t)\big\|_2\le \big\|F_{k+1}\big\|_1\big\|\big(\partial_x M\big)^r M^{2j}\partial_x^{k+1}G_m(t)\big\|_2$$
$$\le C\big\|\,|x|^{1+k}v_0\big\|_1 t^{-\frac{1}{m}(r(m+1)+2mj+k+1)-\frac{1}{2m}}.\qquad □$$

Proof of Theorem 3.3 :

¿From Lemma 3.6 it follows that

$$(3.40)\qquad \Big\|S(t)v_0-\sum_{r=0}^{N}\frac{t^r}{r!}K_m^r*\big(\partial_x M\big)^r S_\varphi(t)*v_0\Big\|_2\le Ct^{-\frac{N+1}{m}-\frac{1}{2m}}\big\|v_0\big\|_1+Ce^{-\frac{t}{2}}\big\|v_0\big\|_2\,.$$

Moreover from Lemma 3.7 we have

$$(3.41)\quad \Big\|\sum_{r=0}^{N}\frac{t^r}{r!}K_m^r*\big(\partial_x M\big)^r S_\varphi*v_0-\sum_{r=0}^{N}\frac{t^r}{r!}\sum_{j=0}^{[\frac{N}{2}]}\frac{t^j}{j!}K_m^{r+j}*\big(\partial_x M\big)^r M^{2j}G_m*v_0\Big\|_2$$
$$\le Ct^{-([\frac{N}{2}]+1)-\frac{1}{2m}}\Big(\sum_{r=0}^{N}\frac{t^{-\frac{r}{m}}}{r!}\Big)\big\|v_0\big\|_1\quad +Ce^{-\frac{t}{2}}t^{-\frac{1}{2m}}\Big(\sum_{r=0}^{N}\frac{t^{-\frac{r}{m}}}{r!}\Big)\Big(\sum_{j=0}^{[\frac{N}{2}]}\frac{t^{-j}}{j!}\Big)\big\|v_0\big\|_1\,.$$

Then from (3.40) and (3.41) it follows that

$$\begin{aligned}
(3.42)\qquad & \Big\| S(t)v_0 \, - \sum_{r=0}^{N} \frac{t^r}{r!} \sum_{j=0}^{[\frac{N}{2}]} \frac{t^j}{j!} K_m^{r+j} * \big(\partial_x M\big)^r M^{2j} G_m(t) * v_0 \Big\|_2 \\
& \le C t^{-\frac{N+1}{m} - \frac{1}{2m}} \big\| v_0 \big\|_1 + \, C t^{-([\frac{N}{2}]+1) - \frac{1}{2m}} \Big(\sum_{r=0}^{N} \frac{t^{-\frac{r}{m}}}{r!} \Big) \big\| v_0 \big\|_1 \\
& \quad + C e^{-\frac{t}{2}} \big\| v_0 \big\|_2 + C e^{-\frac{t}{2}} t^{-\frac{1}{2m}} \Big(\sum_{r=0}^{N} \frac{t^{-\frac{r}{m}}}{r!} \Big) \Big(\sum_{j=0}^{[\frac{N}{2}]} \frac{t^{-j}}{j!} \Big) \big\| v_0 \big\|_1 .
\end{aligned}$$

On the other hand from the Lemma 3.8, we obtain

$$\begin{aligned}
(3.43)\qquad & \Big\| \sum_{r=0}^{N} \frac{t^r}{r!} \sum_{j=0}^{[\frac{N}{2}]} \frac{t^j}{j!} K_m^{r+j} * \big(\partial_x M\big)^r M^{2j} G_m(t) * v_0 \\
& \qquad - \sum_{r=0}^{N} \frac{t^r}{r!} \sum_{j=0}^{[\frac{N}{2}]} \frac{t^j}{j!} \sum_{\alpha=0}^{k(r,j)} \mathcal{M}_\alpha(v_0) K_m^{r+j} * \big(\partial_x M\big)^r M^{2j} \partial_x^\alpha G_m(t) \Big\|_2 \\
& \le \sum_{r=0}^{N} \frac{t^r}{r!} \sum_{j=0}^{[\frac{N}{2}]} \frac{t^j}{j!} \Big\| K_m^{r+j} * \big(\partial_x M\big)^r M^{2j} G_m(t) * v_0 \\
& \qquad - \sum_{\alpha=0}^{k(r,j)} \mathcal{M}_\alpha(v_0) K_m^{r+j} * \big(\partial_x M\big)^r M^{2j} \partial_x^\alpha G_m(t) \Big\|_2 \\
& \le C \sum_{r=0}^{N} \frac{t^r}{r!} \sum_{j=0}^{[\frac{N}{2}]} \frac{t^j}{j!} \, t^{-\frac{1}{m}[k(r,j)+1+r(m+1)+2mj] - \frac{1}{2m}} \big\| |x|^{k(r,j)+1} v_0 \big\|_1 .
\end{aligned}$$

In (3.43), $k(r,j)$ denotes a natural number and we use that $\big\| K_m^r \big\|_1 = 1$.
We choose $k(r,j) = N - r - mj \ge 0$ in (3.43). Hence

$$\begin{aligned}
(3.44)\qquad & \Big\| \sum_{r=0}^{N} \frac{t^r}{r!} \sum_{j=0}^{[\frac{N}{2}]} \frac{t^j}{j!} K_m^{r+j} * (\partial_x M)^r M^{r+j} G_m(t) * v_0 - \\
& - \sum_{r=0}^{N} \frac{t^r}{r!} \sum_{j=0}^{[\frac{N}{2}]} \frac{t^j}{j!} \sideset{}{^*}\sum_{0 \le \alpha \le N-r-mj} \mathcal{M}_\alpha(v_0) K_m^{j} * \big(\partial_x M\big)^r M^{2j} \partial_x^\alpha G_m(t) \Big\|_2 \\
& \le C t^{-\frac{N+1}{m} - \frac{1}{2m}} \sum_{r=0}^{N} \sum_{\substack{0 \le j \le [\frac{N}{2}] \\ 0 \le j \le [\frac{N-r}{m}]}} \big\| |x|^{(N+1)-r-jm} v_0 \big\|_1 .
\end{aligned}$$

Finally, Theorem 3.3 is consequence of (3.42) and (3.44). Indeed,

$$\begin{aligned}
&\Big\| S(t)v_0 - \sum_{r=0}^{N} \frac{t^r}{r!} \sum_{j=0}^{[\frac{N}{2}]} \frac{t^j}{j!} \sum_{0\le\alpha\le N-r-mj}^{*} \mathcal{M}_\alpha(v_0) K_m^{r+j} * \big(\partial_x M\big)^r M^{2j} \partial_x^\alpha G_m(t) \Big\|_2 \\
&\qquad \le Ct^{-\frac{N+1}{m}-\frac{1}{2m}} \|v_0\|_1 + \; Ct^{-([\frac{N}{2}]+1)-\frac{1}{2m}} \Big(\sum_{r=0}^{N} \frac{t^{-\frac{r}{m}}}{r!} \Big) \|v_0\|_1 + \\
&\qquad\quad + Ct^{-\frac{N+1}{m}-\frac{1}{2m}} \sum_{r=0}^{N} \sum_{\substack{0\le j\le[\frac{N}{2}] \\ 0\le j\le[\frac{N-r}{m}]}} \big\| |x|^{N+1-r-jm} v_0 \big\|_1 + \\
&\qquad\quad + Ce^{-\frac{t}{2}} \|v_0\|_2 + Ce^{-\frac{t}{2}} t^{-\frac{1}{2m}} \Big(\sum_{r=0}^{N} \frac{t^{-\frac{r}{m}}}{r!} \Big) \Big(\sum_{j=0}^{[\frac{N}{2}]} \frac{t^{-j}}{j!} \Big) \|v_0\|_1 \, . \qquad \square
\end{aligned}$$

Proof of Theorem 3.4 :

In this case, are continued the same steps of the proof of previous theorem, to consider $N = 1$ and choose $k(r) = 1 - r$ in (3.43). $\square$

4. Complete asymptotic expansion.

The function entering in the $\mathbb{L}^2$-estimate of Theorem 3.3 can be written as

$$\begin{aligned}
(4.1)\quad S(t)v_0 \; - \; &\sum_{\alpha=0}^{N} \mathcal{M}_\alpha(v_0) \partial_x^\alpha G_m(t) \\
&- \sum_{r=0}^{N} \frac{t^r}{r!} \sum_{j=0}^{[\frac{N}{2}]} \frac{t^j}{j!} \sum_{\substack{0\le\alpha\le N-r-mj \\ (r,j)\neq(0,0)}}^{*} \mathcal{M}_\alpha(v_0) \; K_m^{r+j} * \big(\partial_x M\big)^r M^{2j} \partial_x^\alpha G_m(t) \, .
\end{aligned}$$

In this section we simplify the terms of (4.1) that have the form

$$K_m^{r+j} * \big(\partial_x M\big)^r M^{2j} \partial_x^\alpha G_m(t), \tag{4.2}$$

and we replace them, for the first term of their asymptotic expansion, i.e. by

$$(\partial_x M)^r M^{2j} \partial_x^\alpha G_m(t).$$

Now, of the moments formula

$$\int_{\mathbb{R}} x^\alpha f(x) dx = i^\alpha \left(\partial_\xi^\alpha \widehat{f} \right)(0) \; , \; \alpha > 0.$$

Then, observing that

$$\begin{cases} \int_{\mathbb{R}} K_m^j(x) dx = 1, \\ \int_{\mathbb{R}} x K_m^j(x) dx = 0 \; , \quad j \ge 1, m > 1. \end{cases} \tag{4.3}$$

On the other hand, as $K_m^j \in \mathbb{L}^1(\mathbb{R})$, the solution of

$$
\left\{ \begin{array}{l} u_t - Mu = 0 \\ u(x,0) = K_m^j(x)\ , \quad j \geq 1, \end{array} \right. \tag{4.4}
$$

is given by

$$
u(x,t) = \big(K_m^j * G_m(t)\big)(x)\ .
$$

Keeping (4.3) in mind, we have $K_m^j \in \mathbb{L}^1\big(\mathbb{R}, (1+|x|\big)$, since

$$
\int |x|\big|K_m^j(x)\big|dx < \infty.
$$

Consequence of Theorem 3.1 and as $\mathcal{M}_0(K_m^j) = \int K_m^j = 1$, we have the following Corollary:

Corollary 4.1 *There exist a constant $C = C(m) > 0$ such that*

$$
\big\|G_m(t) * K_m^j - G_m(t)\big\|_2 \leq C\, t^{-\frac{1}{m}-\frac{1}{2m}} \big\||x|K_m\big\|_1\ .
$$

Proof.-By Lemma (3.5) with $N = 0$ and $\int K_m(x)dx = 1$ we have

$$
\big\|G_m(t) * K_m - G_m(t)\big\|_2 \leq C\, t^{-\frac{1}{m}-\frac{1}{2m}} \big\||x|K_m\big\|_1\ . \tag{4.5}
$$

Then, from (4.5) we have

$$
\begin{aligned}
\big\|G_m(t) * K_m^2 - G_m(t) * K_m\big\|_2 &\leq \big\|K_m\big\|_1 \big\|G_m(t) * K_m - G_m(t)\big\|_2 \\
&\leq C\, t^{-\frac{1}{m}-\frac{1}{2m}} \big\||x|K_m\big\|_1\ .
\end{aligned} \tag{4.6}
$$

Now, from (4.5) and (4.6), it follows that

$$
\begin{aligned}
\big\|G_m(t) * K_m^2 - G_m(t)\big\|_2 &\leq \big\|G_m(t) * K_m^2 - G_m(t) * K_m\big\|_2 \\
&\quad + \big\|G_m(t) * K_m - G_m(t)\big\|_2 \leq C\, t^{-\frac{1}{m}-\frac{1}{2m}} \big\||x|K_m\big\|_1\ .
\end{aligned}
$$

The conclusion of the proof the Corollary 4.1 it follows by induction. □

If instead of $G_m(t)$ we consider $(\partial_x M)^r M^{2j} \partial_x^\alpha G_m(t)$ in the previous Corollary we obtain the following result:

Corollary 4.2 *Let $r \in \mathbb{N}$. Then there exist a constant $C = C(r,j) > 0$ such that*

$$
\begin{aligned}
\Big\|(\partial_x M)^r M^{2j} \partial_x^\alpha G_m(t) * K_m^{r+j} \; - \; (\partial_x M)^r M^{2j} \partial_x^\alpha G_m(t)\Big\|_2 &\leq \\
&\leq Ct^{-\frac{1+(m+1)r+2jm+|\alpha|}{m}-\frac{1}{2m}} \big\||x|K_m\big\|_1\ .
\end{aligned} \tag{4.7}
$$

And, as an immediate consequence of Corollary 4.2 we have:

Corollary 4.3 *Let $r \in I\!N$. Then there exist a constant $C = C(r,j) > 0$ such that*

$$\left\| \sum_{r=0}^{N} \frac{t^r}{r!} \sum_{j=0}^{[\frac{N}{2}]} \frac{t^j}{j!} \sum_{\substack{0\le\alpha\le N-r-mj \\ (r,j)\neq(0,0)}}^{*} \mathcal{M}_\alpha(v_0)\Big(K_m^{r+j} * \big(\partial_x M\big)^r M^{2j} \partial_x^\alpha G_m(t) - \big(\partial_x M\big)^r M^{2j} \partial_x^\alpha G_m(t) \Big) \right\|_2$$
$$\le C \sum_{r=0}^{N} \frac{t^r}{r!} \sum_{j=0}^{[\frac{N}{2}]} \frac{t^j}{j!} \sum_{0\le\alpha\le N-r-mj}^{*} |\mathcal{M}_\alpha(v_0)|\, t^{-\frac{1+(m+1)r+2jm+\alpha}{m}-\frac{1}{2m}} \big\| |x| K_m \big\|_1$$
$$\le C \max_{0\le\alpha\le N-1} \Big\{ |\mathcal{M}_\alpha(v_0)| \Big\}\, t^{-\frac{1+N}{m}-\frac{1}{2m}} \big\| |x| K_m \big\|_1 .$$

Then the proof of Theorem 2.1 follows directly from Corollary 4.3 and equation (4.1), indeed:

$$\left\| S(t)v_0 \; - \sum_{0\le\alpha\le N} \mathcal{M}_\alpha(v_0)\partial_x^\alpha G_m(t) \; - \; \sum_{r=0}^{N} \frac{t^r}{r!} \sum_{j=0}^{[\frac{N}{2}]} \frac{t^j}{j!} \sum_{\substack{0\le\alpha\le N-r-mj \\ (r,j)\neq(0,0)}}^{*} \mathcal{M}_\alpha(v_0)\big(\partial_x M\big)^r M^{2j}\partial_x^\alpha G_m(t) \right\|_2$$
$$\le C \Bigg[t^{-([\frac{N}{2}]+1)-\frac{1}{2m}} \big\|v_0\big\|_1 \Big(\sum_{r=0}^{N} \frac{t^{-\frac{r}{m}}}{r!} \Big) + t^{-\frac{N+1}{m}-\frac{1}{2m}} \big\|v_0\big\|_1 + t^{-\frac{N+1}{m}-\frac{1}{2m}} \sum_{r=1}^{N+1} \big\| |x|^r v_0 \big\|_1$$
$$+ \; e^{-\frac{t}{2}} \big\|v_0\big\|_2 \; + \; e^{-\frac{t}{2}} t^{-\frac{1}{2m}} \Big(\sum_{r=0}^{N} \frac{t^{-\frac{r}{m}}}{r!} \Big) \Big(\sum_{j=0}^{[\frac{N}{2}]} \frac{t^{-j}}{j!} \Big) \big\|v_0\big\|_1$$
$$+ \max_{0\le\alpha\le N-1} \Big\{ |\mathcal{M}_\alpha(v_0)| \Big\}\, t^{-\frac{1+N}{m}-\frac{1}{2m}} \big\| |x| K_m \big\|_1 \Bigg] \qquad \square$$

Remark 4.4 *When $m = 2$ the Theorem 2.1 it agree with results in [27]*
Remembering that $\mathcal{M}_\alpha(v_0) = \frac{(-1)^\alpha}{\alpha!} \int x^\alpha v_0(x)dx$. In the Theorem 2.1 we have the following particular cases when $m \in Z^+$, $m > 1$:

1. *If $N = 0$, the firs term in the asymptotic expansion is $r_1(x,t) = \mathcal{M}_0(v_0)\, G_m(t)$. Then, for $v_0 \in I\!L^2(I\!R) \cap I\!L^1(I\!R, 1+|x|)$, we have*

$$t^{\frac{1}{2m}} \big\| S(t)v_0 - \mathcal{M}_0(v_0) G_m(t) \big\|_2 \le C t^{-\frac{1}{m}}, \quad \textit{for} \quad t \ge 1 .$$

2. *If $N = 1$, the firs term in the asymptotic expansion is $r_1(x,t) = \mathcal{M}_0(v_0)G_m(t)$, and the secon term is $r_2(x,t) = \mathcal{M}_1(v_0)\partial_x G_m(t) + t\mathcal{M}_0(v_0)(\partial_x M)G_m(t)$.*
Then for $v_0 \in I\!L^2(I\!R) \cap I\!L^1(I\!R, 1+|x|^2)$ we have

$$t^{\frac{1}{2m}+\frac{1}{m}} \left\| S(t)v_0 - \sum_{j=1}^{2} r_j(t) \right\|_2 \le C t^{-\frac{1}{m}}, \quad \textit{for} \quad t \ge 1 .$$

3. *If $N = 2$, $m = 2$ then, $Mu = -\partial_{xx} u$ and $G_2(x,t) = G(x,t) = (4\pi t)^{-\frac{1}{2}} e^{-\frac{x^2}{4t}}$ it is the heat kernel. The firs term in the asymptotic expansion is*

$r_1(x,t) = \mathcal{M}_0(v_0)G(t)$, the secon term is
$r_2(x,t) = \mathcal{M}_1(v_0)\partial_x G(t) - t\mathcal{M}_0(v_0)\partial_{xxx}G(t)$. And the third one is
$r_3(x,t) = \mathcal{M}_2(v_0)\partial_{xx}G + t\Big(\mathcal{M}_0(v_0) - \mathcal{M}_1(v_0)\Big)\partial_x^4 G + \frac{t^2}{2}\mathcal{M}_0(v_0)\partial_x^6 G$.
Then for $v_0 \in \mathbb{L}^2(\mathbb{R}) \cap \mathbb{L}^1(\mathbb{R}, 1+|x|^3)$ we have

$$t^{\frac{1}{4}+1}\left\| S(t)v_0 - \sum_{j=1}^{3} r_j(t)\right\|_2 \le Ct^{-\frac{1}{2}}, \quad \textit{for} \quad t \ge 1.$$

.

When $m \ge 3$, $N = 2$, the firs term in the asymptotic expansion is
$r_1(x,t) = \mathcal{M}_0(v_0)G_m(t)$, the secon term is
$r_2(x,t) = \mathcal{M}_1(v_0)\partial_x G_m(t) + t\mathcal{M}_0(v_0)(\partial_x M)G_m(t)$. And the third one is
$r_3(x,t) = \mathcal{M}_2(v_0)\partial_{xx}G_m(t) + t\mathcal{M}_1(v_0)(\partial_x M)\partial_x G_m(t) + \frac{t^2}{2}\mathcal{M}_0(v_0)(\partial_x M)^2 G_m(t)$.
Then for $v_0 \in \mathbb{L}^2(\mathbb{R}) \cap \mathbb{L}^1(\mathbb{R}, 1+|x|^3)$ we have

$$t^{\frac{1}{2m}+\frac{2}{m}}\left\| S(t)v_0 - \sum_{j=1}^{3} r_j(t)\right\|_2 \le Ct^{-\frac{1}{m}}, \quad \textit{for} \quad t \ge 1, \quad m \ge 3.$$

Remark 4.5 *When $m = n + \delta$ for $n \in Z^+$, $n > 1$, $0 < \delta < 1$, as consequence of Theorem 2.1, we have*

$$t^{\frac{1}{m}+\frac{1}{2m}}\left\| S(t)v_0 - \mathcal{M}_0(v_0)G_m(t) - \mathcal{M}_1(v_0)\partial_x G_m(t) - t\mathcal{M}_0(v_0)\big(\partial_x M\big)G_m(t)\right\|_2 \le Ct^{-\frac{1}{m}},$$

for $t \ge 1$ and for all $v_0 \in \mathbb{L}^2(\mathbb{R}) \cap \mathbb{L}^1(\mathbb{R}^n; 1+|x|^2)$.

5. Complete asymptotic expansion of the generalized KdV linear equation

Let us consider now the linearized equation

$$\begin{cases} u_t + Mu - Mu_x = 0, & x \in \mathbb{R}, t > 0 \\ u(x,0) = u_0(x)\,. \end{cases} \tag{5.1}$$

Where $\widehat{Mu}(\xi) = |\xi|^m \widehat{u}(\xi)$, $m \in \mathbb{Z}^+$. Its solution takes the form

$$u(x,t) = T(\cdot,t) * u_0(x)\,,$$

with

$$T(x,t) = \frac{1}{2\pi}\int e^{t\psi(\xi)+ix\cdot\xi}d\xi.$$

The phase function is now

$$\psi(\xi) = -|\xi|^m + i\xi|\xi|^m$$

which is a bit simpler than the phase function $\widetilde{\Phi}$ of the (2.4) linear equation. Proceeding as in the proof of Theorem 3.3, Section 3 we obtain the following result:

Theorem 5.1 *For any $N \in I\!N$, there exists a constant $C = C(N) > 0$ such that*

$$\left\| T(t) * u_0 - \sum_{j=0}^{N} \frac{t^j}{j!} \big(\partial_x M\big)^j \sum_{\alpha=0}^{N-j} \mathcal{M}_\alpha(u_0) \partial_x^\alpha G_m(t) \right\|_2$$

$$\leq C\, t^{-\frac{1}{2m} - \frac{N+1}{m}} \big\| u_0 \big\|_1 + C\, t^{-\frac{1}{2m} - \frac{N+1}{m}} \sum_{k=1}^{N+1} \big\| |x|^k u_0 \big\|_1 ,$$

for all $t > 0$ and $u_0 \in I\!L^1(I\!R, 1 + |x|^{N+1})$.

Remark 5.2 *For instance, if $N = 2$, in Theorem 5.1 we have that the first term is $r_1(x,t) = MG_m(t)$. The second term is $r_2(x,t) = \mathcal{M}_1(u_0)G_m(t) + t\mathcal{M}_0(u_0)\big(\partial_x M\big)G_m(t)$, and the third one*

$$r_3(x,t) = \mathcal{M}_2(u_0)\partial_{xx}G_m(t) + t\mathcal{M}_1(u_0)\big(\partial_x M\big)\partial_x G_m(t) + \frac{t^2}{2}\mathcal{M}_0(u_0)\big(\partial_x M\big)^2 G_m(t).$$

Then for $u_0 \in I\!L^1(I\!R, 1 + |x|^3)$ we have

$$t^{\frac{1}{4}+1} \left\| T(t)u_0 - \sum_{j=1}^{3} r_j(t) \right\|_2 \leq C t^{-\frac{1}{m}} , \quad \text{for} \quad t \geq 1 .$$

Remark 5.3 *In accordance with Theorems 2.1 and 5.1, the succesive terms appearing in the asymptotic expansion of the solutions of (2.4) and (5.1) have the form*

$$\sum_{r=0}^{N} \frac{t^r}{r!} \sum_{j=0}^{[\frac{N}{2}]} \frac{t^j}{j!} \big(\partial_x M\big)^r M^{2j} \sum_{\alpha=0}^{N-r-2j} \mathcal{M}_\alpha(v_0)\partial_x^\alpha G_m(t) \tag{5.2}$$

and

$$\sum_{k=0}^{N} \frac{t^k}{k!} \big(\partial_x M\big)^k \sum_{\alpha=0}^{N-k} \mathcal{M}_\alpha(u_0)\partial_x^\alpha G_m(t) , \tag{5.3}$$

respectively. We see that, the term due to dispersive effects of Mu_x in (5.1) and Mu_x and Mv_t in (2.4), appear in the asymptotic expansions starting at the second term.

6 Global Solution

In this section we study the global solution of the initial value problem for the following model equation (2.3) and suppose that $m > 2$. There is a well known principle which has frequently been used to prove existence of global solution of non-linear equations. Indeed, we may rewrite our non-linear diferential partial equation as non-linear equation integral, obtained from the formula of variation of the parameters (or formula of Duhamel)

$$v(x,t) = S(t)v_0(x) - \int_0^t S(t-\tau)K_m * \big(v^q\big)_x(\tau) d\tau. \tag{6.4}$$

Where

$$(6.5) \qquad S(t)v_0(x) = \frac{1}{(2\pi)} \int \exp\big(- t\Phi(\xi) + ix \cdot \xi \big) \widehat{v_0}(\xi) d\xi \, ,$$

with the phase function

$$\Phi(\xi) = \frac{|\xi|^m - i\xi|\xi|^m}{1 + |\xi|^m}.$$

Recall that K_m is the function defined by $\widehat{K}(\xi) = \frac{1}{1+|\xi|^m}$.
The equation (6.4) is equivalent to the differential form (2.3), but is much easier to handle when it comes to proving questions of existence an uniqueness.

To find a solution v to (6.4), we shall use an iterative method. We first approximate v by the linear solution

$$(6.6) \qquad v_0(x,t) = S(t)v_0(x).$$

Then we make a better approximation

$$v_1(x,t) = S(t)v_0(x) - \int_0^t S(t-\tau) K_m * \big(v_0^q\big)_x(\tau) d\tau.$$

More generally, we define the non-linear map $v \mapsto Pv$ by

$$Pv(x,t) = S(t)v_0(x) - \int_0^t S(t-\tau) K_m * \big(v^q\big)_x(\tau) d\tau$$

and define $v_{k+1} = P(v_k)$ for all $k = 0, 1, \ldots.$ We hope to show that this sequence of approximations converges to a limit v, so that $v = Pv$. This would give us a solution to (6.4).

In short, we want to find a fixed point of P, show that it is unique. We can accomplish all this in one stroke from the Banachs fixed point Theorem, as soon as we show that P is a contraction on some complete metric space X which contains v_0. Well, we have to pick the right complete metric space to get the contraction working.

Theorem 6.1 *Suppose that $v_0 \in W^{2,1}(I\!R)$; then there exist a positive constant δ_1 such that when $\|v_0\|_{W^{2,1}} < \delta_1$, then the equation (6.4), have a unique global solution $v(x,t)$ satisfying*

$$v(x,t) \in C(0,\infty; I\!L^\infty(I\!R) \cap H^2(I\!R)).$$

Moreover, the asymptotic decay rates of the solutions $v(x,t)$, (2.6), holds.

Initial data v_0 for which $\|v_0\|_{W^{2,1}}$ is sufficiently small give rise to global solution of the nonlinear equation.

6.1 Properties of the linear solutions $S(t)v_0$

Lemma 6.2 *Let $v_0 \in W^{2,1}(I\!R)$. Then there exist a positive constant C, independent of t such that*

$$\big\|S(t)v_0\big\|_2 \le C\|v_0\|_{W^{1,1}}(1+t)^{-\frac{1}{2m}} \tag{6.7}$$

$$\big\|S_x(t)v_0\big\|_2 \le C\|v_0\|_{W^{2,1}}(1+t)^{-\frac{1}{2m}-\frac{1}{m}} \tag{6.8}$$

$$\big\|S(t)v_0\big\|_\infty \le C\|v_0\|_{W^{2,1}}(1+t)^{-\frac{1}{m}}. \tag{6.9}$$

Proof.-By Plancherel's formula, we have

$$\big\|S(t)v_0\big\|_2^2 = \int \frac{\mathrm{e}^{-\frac{2t|\xi|^m}{1+|\xi|^m}}}{1+|\xi|^2}(1+|\xi|^2)|\widehat{v_0}(\xi)|^2 d\xi \le \Big[\sup_{\xi\in I\!R}\Big\{(1+|\xi|)|\widehat{v_0}(\xi)|\Big\}\Big]^2 \int_{I\!R} \frac{\mathrm{e}^{-\frac{2t|\xi|^m}{1+|\xi|^m}}}{1+|\xi|^2}. \tag{6.10}$$

Now

$$\int_{I\!R} \frac{\mathrm{e}^{-\frac{2t|\xi|^m}{1+|\xi|^m}}}{1+|\xi|^2} \le 2\Big(\int_0^1 + \int_1^\infty\Big). \tag{6.11}$$

If $\xi \in [0,1]$ then $\frac{1}{2} \le \frac{1}{1+|\xi|^m} \le 1$, hence $\mathrm{e}^{-\frac{2t|\xi|^m}{1+|\xi|^m}} \le \mathrm{e}^{-t|\xi|^m}$, we have

$$\begin{aligned}\int_0^1 \frac{\mathrm{e}^{\frac{-2t|\xi|^m}{1+|\xi|^m}}}{1+|\xi|^2}d\xi \le& C\int_0^1 \mathrm{e}^{-t|\xi|^m}d\xi \le C\int_0^1 \mathrm{e}^{-(1+t)|\xi|^m}\mathrm{e}^{|\xi|^m}d\xi \\ &\le C\int_0^1 \mathrm{e}^{-(1+t)|\xi|^m}d\xi \le C(1+t)^{-\frac{1}{m}}.\end{aligned} \tag{6.12}$$

For the second term on the right-hand side of (6.11), if $\xi \in [1,\infty)$ then $\frac{1}{2} \le \frac{|\xi|^m}{1+|\xi|^m} \le 1$, hence

$$\int_1^\infty \frac{\mathrm{e}^{\frac{-2t|\xi|^m}{1+|\xi|^m}}}{1+|\xi|^2}d\xi \le \int_1^\infty \frac{\mathrm{e}^{-t}}{1+|\xi|^2}d\xi \le C\mathrm{e}^{-t}. \tag{6.13}$$

On the other hand

$$\begin{aligned}\sup_{\xi\in I\!R}\Big\{(1+|\xi|)|\widehat{v_0}(\xi)|\Big\} &\le \sup_{\xi\in I\!R}\Big|\int e^{-ix\cdot\xi}v_0(x)dx\Big| + \sup_{\xi\in I\!R}\Big|\int e^{-ix\cdot\xi}v_{0x}(x)\Big|dx \\ &\le \int|v_0(x)|dx + \int|v_{0x}(x)|dx \le \|v_0\|_{W^{1,1}}.\end{aligned} \tag{6.14}$$

Thus, plugging (6.12), (6.13) and (6.14) into (6.10) implies (6.7).

Proof of (6.8), we have similarly

$$\begin{aligned}\big\|S_x(t)v_0\big\|_2^2 &= \int \frac{(1+|\xi|^2)^2|\xi|^2\mathrm{e}^{-\frac{2t|\xi|^m}{1+|\xi|^m}}}{(1+|\xi|^2)^2}|\widehat{v_0}(\xi)|^2 d\xi \\ &\le \Big[\sup_{\xi\in I\!R}\Big\{(1+|\xi|^2)|\widehat{v_0}(\xi)|\Big\}\Big]^2\int_{I\!R}\frac{|\xi|^2\mathrm{e}^{-\frac{2t|\xi|^m}{1+|\xi|^m}}}{(1+|\xi|^2)^2}.\end{aligned} \tag{6.15}$$

Now,

$$\int_{I\!R} \frac{|\xi|^2 \mathrm{e}^{-\frac{2t|\xi|^m}{1+|\xi|^m}}}{(1+|\xi|^2)^2} \leq 2\Big(\int_0^1 + \int_1^\infty \Big).$$

Then, $\xi \in [0,1]$ implies that $\frac{1}{2} \leq \frac{1}{1+|\xi|^m} \leq 1$, we get

$$(6.16)\quad \int_0^1 |\xi|^2 \mathrm{e}^{\frac{-2t|\xi|^m}{1+|\xi|^m}} d\xi \leq C \int_0^1 |\xi|^2 \mathrm{e}^{-(1+t)|\xi|^m} \mathrm{e}^{|\xi|^m} d\xi \leq C \int_0^1 |\xi|^2 \mathrm{e}^{-(1+t)|\xi|^m} d\xi \leq C(1+t)^{-\frac{2}{m}-\frac{1}{m}}.$$

Using the facts that $\frac{1}{2} \leq \frac{|\xi|^m}{1+|\xi|^m} < 1$ if $\xi \in [1,\infty)$, we have

$$(6.17)\quad \int_1^\infty \frac{|\xi|^2 \mathrm{e}^{-\frac{2t|\xi|^m}{1+|\xi|^m}}}{(1+|\xi|^2)^2} d\xi \leq \int_1^\infty \frac{\mathrm{e}^{-t}}{1+|\xi|^2} d\xi \leq C\mathrm{e}^{-t}.$$

Also,

$$(6.18)\quad \begin{aligned} \sup_{\xi \in I\!R} \Big\{ (1+|\xi|^2)|\widehat{v_0}(\xi)| \Big\} &\leq \sup_{\xi \in I\!R} \left| \int e^{-ix\cdot\xi} v_0(x) dx \right| + \sup_{\xi \in I\!R} \left| \int e^{-ix\cdot\xi} v_{0xx}(x) \right| dx \\ &\leq \int |v_0(x)| dx + \int |v_{0xx}(x)| dx \leq \|v_0\|_{W^{2,1}}. \end{aligned}$$

Therefore (6.16), (6.17) and (6.18) give us the desired estimate (6.8).

Proof of (6.9), using the classical inequality

$$\big\|S(t)v_0\big\|_\infty \leq \big\|S(t)v_0\big\|_2^{1/2} \big\|S_x(t)v_0\big\|_2^{1/2},$$

(6.7) and (6.8), we get (6.9). □

Now, recall that K_m is the function defined by $\widehat{K_m}(\xi) = 1/(1+|\xi|^m)$, we have:

Lemma 6.3 *Let $m > 3/2$. If*

$$S(t)K_m(x) = \frac{1}{(2\pi)} \int exp\big(-t\widetilde{\Phi}(\xi) + ix\cdot\xi \big) \widehat{K_m}(\xi) d\xi,$$

with the phase function

$$\widetilde{\Phi}(\xi) = \frac{|\xi|^m - i\xi|\xi|^m}{1+|\xi|^m}.$$

Then, there exist a positive constant C, independent of t, such that

$$(6.19)\quad \big\|S(t)K_m\big\|_2 \leq C(1+t)^{-\frac{1}{2m}};$$

$$(6.20)\quad \big\|S_x(t)K_m\big\|_2 \leq C(1+t)^{-\frac{1}{2m}-\frac{1}{m}};$$

$$(6.21)\quad \big\|S(t)K_m\big\|_\infty \leq C(1+t)^{-\frac{1}{m}}.$$

Proof.-By Plancherel formula we have

$$\begin{aligned} \big\|S(t)K_m\big\|_2^2 = &\big\|\widehat{S(t)K_m}\big\|_2^2 = \int \frac{\mathrm{e}^{-\frac{2t|\xi|^m}{1+|\xi|^m}}}{(1+|\xi|^m)^2} d\xi = \int_0^1 + \int_1^\infty \\ &\leq \int_0^1 \frac{\mathrm{e}^{-t|\xi|^m}}{(1+|\xi|^m)^2} d\xi + \int_1^\infty \frac{\mathrm{e}^{-t}}{(1+|\xi|^m)^2} d\xi \leq C(1+t)^{-1/m} + Ce^{-t}. \end{aligned}$$

This inequality implies (6.19).

Similarly, we have

$$\big\|S_x(t)K_m\big\|_2^2 = \big\|\widehat{S_x(t)K_m}\big\|_2^2 = \int \frac{|\xi^2|\mathrm{e}^{-\frac{2t|\xi|^m}{1+|\xi|^m}}}{(1+|\xi|^m)^2}d\xi = \int_0^1 + \int_1^\infty$$

$$\leq \int_0^1 \frac{|\xi|^2\mathrm{e}^{-t|\xi|^m}}{(1+|\xi|^m)^2}d\xi + \int_1^\infty \frac{\mathrm{e}^{-t}\ |\xi|^2}{(1+|\xi|^m)^2}d\xi \leq C(1+t)^{-\frac{2}{m}-\frac{1}{m}} + Ce^{-t}, \ \ (m>3/2).$$

Hence, (6.20) it is proven.

Now,

$$\big|S(t)K_m\big| \leq C\int \frac{\mathrm{e}^{-\frac{t|\xi|^m}{1+|\xi|^m}}}{1+|\xi|^m}d\xi \leq C\int_0^1 \mathrm{e}^{-\frac{t|\xi|^m}{2}}d\xi + C\int_1^\infty \frac{\mathrm{e}^{-\frac{t}{2}}}{1+|\xi|^m}d\xi \leq C(1+t)^{-1/m} + Ce^{-\frac{t}{2}}.$$

Then, (6.21) it is proven. □

6.2 Proof of Theorem 6.1

Looking at the behaviour of the first few iterates v_0, v_1, etc., by the Lema 6.2, we decide that the correct metric space to use is (as in [15, 25] for example)

$$X_\delta = \Big\{v \in C\big(0,\infty; H^2 \cap I\!\!L^\infty\big)\ /M(v) < \delta\Big\}, \qquad m > 2\ .$$

Endowed with the distance

$$M(v) = \sup_{0\leq t<\infty}\Big\{(1+t)^{\frac{1}{2m}}\big\|v(t)\big\|_2 + (1+t)^{\frac{1}{m}}\big\|v(t)\big\|_\infty + (1+t)^{\frac{1}{2m}+\frac{1}{m}}\big\|v_x(t)\big\|_2\Big\},$$

X_δ is a complete metric space. We will show that the mapping defined formally by

$$P(v) = S(t)v_0(x) - \int_0^t S(t-\tau)K_m * \big(v^q\big)_x(\tau)d\tau,$$

is a strict contraction on X_δ, that is, we will prove that there exist the positive constant δ_1, such that the operator P maps X_{δ_1} into itself and has a unique fixed point in X_{δ_1}. Thus, such a fixed point $v(x,t)$ is the unique solution of equation (6.4) in X_{δ_1}, globally in time. To prove these, the following Lemmas are avaliable.

Lemma 6.4 *Suppose that $a>0$ and $b>0$ and* $\max\{a,b\}>1$, *then*

$$\int_0^t (1+t-\tau)^{-a}(1+\tau)^{-b}d\tau \leq (1+t)^{-\min\{a,b\}}. \tag{6.22}$$

The Lemma 6.4 can be found in [31], see also [24].

Lemma 6.5 *Let $q>1$. Then there exists a constant $C>0$, independent of t, such that*

$$\big\||u(t)|^{q-1}u(t) - |v(t)|^{q-1}v(t)\big\|_r \leq C\big\|u(t)-v(t)\big\|_{p_1}\bigg(\big\|u(t)\big\|_{(q-1)p_2}^{q-1} + \big\|v(t)\big\|_{(q-1)p_2}^{q-1}\bigg), \tag{6.23}$$

for $\frac{1}{r} = \frac{1}{p_1} + \frac{1}{p_2} \leq 1$.

Proof.-If $\alpha > 0$, we obtain

$$|a|^\alpha a - |b|^\alpha b = (1+\alpha)(a-b)\int_0^1 |\tau(a-b)+b|^\alpha d\tau,$$

for all $a, b \in I\!R$. In our case we have that

$$|u(t)|^\alpha u(t) - |v(t)|^\alpha v(t) = (1+\alpha)(u(t)-v(t))\int_0^1 |\tau(u(t)-v(t))+v(t)|^\alpha d\tau \ .$$

Using the Holder inequality for $\frac{1}{r} = \frac{1}{p_1} + \frac{1}{p_2} \leq 1$, we obtain

$$\begin{aligned}
\| \, |u(t)|^{q-1}u(t) - |v(t)|^{q-1}v(t)\|_r &\leq C\|u(t)-v(t)\|_{p_1}\int_0^1 \|(\tau(u-v)+v)^{q-1}\|_{p_2} d\tau \\
&\leq C\|u(t)-v(t)\|_{p_1}\int_0^1 \|\tau(u-v)+v\|^{q-1}_{(q-1)p_2} d\tau \\
&\leq C\|u(t)-v(t)\|_{p_1}\left(\|u(t)\|^{q-1}_{(q-1)p_2} + \|v(t)\|^{q-1}_{(q-1)p_2}\right) . \quad \square
\end{aligned}$$

Lemma 6.6 *Let $q > m$ and $m > 2$. If $v \in X_\delta$, we have*

$$\int_0^t \left\| S(t-\tau)K_m * \left(v^q\right)_x(\tau)\right\|_2 d\tau \leq C\delta^q(1+t)^{-\frac{1}{2m}}; \tag{6.24}$$

$$\int_0^t \left\| S_x(t-\tau)K_m * \left(v^q\right)_x(\tau)\right\|_2 d\tau \leq C\delta^q(1+t)^{-\frac{1}{2m}-\frac{1}{m}}; \tag{6.25}$$

$$\int_0^t \left\| S_x(t-\tau)K_m * \left(v^q\right)_x(\tau)\right\|_\infty d\tau \leq C\delta^q(1+t)^{-\frac{1}{m}}. \tag{6.26}$$

Proof.-Note that

$$\begin{aligned}
\left\|v^{(q-1)}v_x(t)\right\|_1 \leq& \left\|v^{(q-1)}(t)\right\|_2\left\|v_x(t)\right\|_2 \leq \left\|v(t)\right\|^{q-m}_\infty\left\|v(t)\right\|^{m-1}_{2(m-1)}\left\|v_x(t)\right\|_2 \\
&\leq \delta^q \ (1+\tau)^{-\frac{q-m}{m}-\frac{m-1}{m}(1-\frac{1}{2(m-1)})-\frac{1}{2m}-\frac{1}{m}} \leq \delta^q(1+\tau)^{-\frac{q}{m}}.
\end{aligned} \tag{6.27}$$

Proof of (6.24), by (6.19) and (6.27) we have

$$\begin{aligned}
\int_0^t \left\|S(t-\tau)K_m * \left(v^q\right)_x(\tau)\right\|_2 d\tau &\leq \int_0^t \left\|S(t-\tau)K_m\right\|_2\left\|v^{(q-1)}v_x(\tau)\right\|_1 d\tau \\
&\leq C\delta^q\int_0^t (1+t-\tau)^{-\frac{1}{2m}}(1+\tau)^{-\frac{q}{m}} d\tau \leq \delta^q(1+t)^{-\frac{1}{2m}}.
\end{aligned}$$

In the ultima line we use that $q > m$ and 6.22.
Proof of (6.25), by (6.20), (6.22) and (6.27) we have

$$\begin{aligned}
\int_0^t \left\|S_x(t-\tau)K_m * \left(v^q\right)_x(\tau)\right\|_2 d\tau &\leq \int_0^t \left\|S_x(t-\tau)K_m\right\|_2\left\|v^{(q-1)}v_x(\tau)\right\|_1 d\tau \\
&\leq\delta^q\int_0^t (1+t-\tau)^{-\frac{1}{2m}-\frac{1}{m}}(1+\tau)^{-\frac{q}{m}} d\tau \leq C\delta^q(1+t)^{-\frac{1}{2m}-\frac{1}{m}}.
\end{aligned}$$

Proof of (6.26), by (6.21), (6.27) and (6.22) we have

$$\int_0^t \Big\|S(t-\tau)K_m * \big(v^q\big)_x(\tau)\Big\|_\infty \, d\tau \le \int_0^t \big\|S(t-\tau)K_m\big\|_\infty \big\|v^{(q-1)}v_x(\tau)\big\|_1 \, d\tau$$
$$\le \delta^q \int_0^t (1+t-\tau)^{-\frac{1}{m}}(1+\tau)^{-\frac{q}{m}} \, d\tau \le C\delta^q(1+t)^{-\frac{1}{m}}. \quad \square$$

For the contraction property of P, we need the following Lema:

Lemma 6.7 *Let $u(x,t), v(x,t) \in X_\delta$. Then there exist a positive constant C, independent of t, such that*

$$\big\|\big(u^q\big)_x - \big(v^q\big)_x\big\|_1 \le C\delta^q M(u-v)(1+t)^{-q/m}, \qquad q > m.$$

Proof.-We have that $|u|^{q-1}u_x - |v|^{q-1}v_x = |u|^{q-1}\big(u_x - v_x\big) + v_x\big(|u|^{q-1} - |v|^{q-1}\big)$, hence

$$\begin{aligned}
(6.28)\quad \big\||u|^{q-1}u_x - |v|^{q-1}v_x\big\|_1 &\le \big\|u^{q-1}\big\|_2 \, \big\|u_x - v_x\big\|_2 + \big\|v_x\big\|_2 \, \big\||u|^{q-1} - |v|^{q-1}\big\|_2 \\
&\le \big\|u\big\|_\infty^{q-2} \, \big\|u\big\|_2 \, \big\|u_x - v_x\big\|_2 + \big\|v_x\big\|_2 \Big[\big\|u-v\big\|_2\Big(\big\|u\big\|_\infty^{q-1} + \big\|v\big\|_\infty^{q-1}\Big)\Big] \ \text{(by (6.23))} \\
&\le \delta^{q-1}\big\|u_x - v_x\big\|_2 \, (1+t)^{-\frac{q-2}{m}-\frac{1}{2m}} + \delta^q\big\|u-v\big\|_2 \, (1+t)^{-\frac{1}{2m}-\frac{1}{m}-\frac{q-1}{m}} \\
&= \delta^{q-1}(1+t)^{\frac{1}{2m}+\frac{1}{m}} \, (1+t)^{-\frac{q}{m}} + \delta^q(1+t)^{\frac{1}{2m}}\big\|u-v\big\|_2 \, (1+t)^{-\frac{q}{m}-\frac{1}{m}} \\
&\le \delta^{q-1}M(u-v)(1+t)^{-\frac{q}{m}} + \delta^q M(u-v)(1+t)^{-\frac{q}{m}-\frac{1}{m}} \\
&\le \delta^q M(u-v)(1+t)^{-\frac{q}{m}}. \qquad \square
\end{aligned}$$

To prove Theorem 1, we need to prove that there exists the positive constant δ, such that the operator P is a contraction mapping from X_{δ_1} into X_{δ_1}.

Step 1. $P : X_\delta \to X_\delta$. For any $v_1(x,t) \in X_\delta$, and denoting $v = Pv_1$, we will prove that $v = Pv_1 \in X_\delta$ for some small $\delta > 0$.

Indeed, using (6.7) and (6.24) we have

$$(6.29)\qquad \big\|v(t)\big\|_2 = \big\|Pv_1(t)\big\|_2 \le \big\|S(t)v_0\big\|_2 + \int_0^t \big\|S(t-\tau)K_m * (v_1^q)_x(\tau)\big\|_2 d\tau$$
$$\le C\big\|v_0\big\|_{W^{1,1}}(1+t)^{-\frac{1}{2m}} + C\delta^q(1+t)^{-\frac{1}{2m}}.$$

Similarly, we have due to (6.8) and (6.25)

$$(6.30)\qquad \big\|v_x(t)\big\|_2 \le C\big\|v_0\big\|_{W^{2,1}}(1+t)^{-\frac{1}{2m}-\frac{1}{m}} + C\delta^q(1+t)^{-\frac{1}{2m}-\frac{1}{m}}.$$

By the same way, we can prove that

$$(6.31)\qquad \big\|v(t)\big\|_\infty \le C\big\|v_0\big\|_{W^{2,1}}(1+t)^{-\frac{1}{m}} + C\delta^q(1+t)^{-\frac{1}{m}}.$$

Thus, combining (6.29), (6.30) and (6.31) implies that

$$M(v) \le C_1\big(\big\|v_0\big\|_{W^{2,1}} + \delta^q\big), \quad q > m > 3/2.$$

Then there exist some small $\delta_2 > 0$, such that $\delta_2^{q-1} < \frac{1}{2C_1}$. Let $\|v_0\|_{W^{2,1}} \leq \frac{\delta_2}{2C_1}$, and $\delta \leq \delta_2$, then

$$M(v) \leq C_1\Big(\frac{\delta_2}{2C_1} + \delta_2^{q-1}\delta_2\Big) < \frac{\delta_2}{2} + \frac{\delta_2}{2} = \delta_2.$$

We have proved $M(v) \leq \delta$ for some small δ, namely, $P : X_\delta \to X_\delta$ for some small $\delta < \delta_2$.

Step 2. P is a contraction in X_δ. Let $u, v \in X_\delta$, from Lema 6.7 and Young inequality it follows that

(6.32)

$$\begin{aligned}\|Pu - Pv\|_2 &\leq \int_0^t \|S(t-\tau)K_m\|_2 \, \|(u^q)_x(\tau) - (v^q)_x(\tau)\|_1 d\tau \\ &\leq C\delta^q M(u-v)\int_0^t (1+t-\tau)^{-\frac{1}{2m}}(1+\tau)^{-\frac{q}{m}}d\tau \leq C\delta^q M(u-v)(1+t)^{-\frac{1}{2m}}.\end{aligned}$$

We have, in the same way in (6.32)

$$\begin{aligned}\|(Pu - Pv)_x\|_2 &\leq \int_0^t \|S_x(t-\tau)K_m\|_2 \, \|(u^q)_x(\tau) - (v^q)_x(\tau)\|_1 d\tau \\ &\leq C\delta^q M(u-v)\int_0^t (1+t-\tau)^{-\frac{1}{2m}-\frac{1}{m}} \,(1+\tau)^{-\frac{q}{m}}d\tau \leq C\delta^q M(u-v)(1+t)^{-\frac{1}{2m}-\frac{1}{m}}.\end{aligned} \tag{6.33}$$

And also

$$\begin{aligned}\|Pu - Pv\|_\infty &\leq \int_0^t \|S(t-\tau)K_m\|_\infty \, \|(u^q)_x(\tau) - (v^q)_x(\tau)\|_1 d\tau \\ &\leq C\delta^q M(u-v)\int_0^t (1+t-\tau)^{-\frac{1}{m}}(1+\tau)^{-\frac{q}{m}}d\tau \leq C\delta^q M(u-v)(1+t)^{-\frac{1}{m}}.\end{aligned} \tag{6.34}$$

Therefore, from (6.32), (6.33) and (6.34), we obtain

$$M(Pu - Pv) \leq C\delta^q M(u-v) \tag{6.35}$$

Let choose $\delta \leq \delta_3 < \frac{1}{C^{1/q}}$; we have proved

$$M(Pu - Pv) < M(u-v),$$

i.e. $P : X_\delta \to X_\delta$ is a contraction for some small $\delta < \delta_3$. Thank to steps 1 and 2, let $\delta_1 < \min\{\delta_2, \delta_3\}$, we have proved that the operator P is contraction from X_{δ_1} to X_{δ_1}. By the Banach's fixed point Theorem, we see that P has a unique fixed point $v(x,t)$ in X_{δ_1}. This means that the integral equation (6.4) has a unique global solution $v(x,t) \in X_{\delta_1}$. Thus, we have completed the proof of Theorem. □

7. Asymptotic expansion: non-linear case

In this section we prove some preliminary results which leads to the prove of the Theorem 2.4, i.e. to the asymptotic expansion of the solutions of the equation (2.3), where $q > m$, and $m > 2$.

The solution of (2.3) satisfies the integral equation (6.4) obtained from the variation of constants formula. It is also convenient to recall that the solution of (2.3) satisfy the decay properties (2.6).

We now prove some preliminary results.

Noting that, from Theorem 3.1, with $N = 0$ and $\mathcal{M} = \int_{I\!R} u_0(x)dx$, we have

$$t^{\frac{1}{m}(1-\frac{1}{p})}\big\|G_m(t) * u_0 - \mathcal{M}G_m(t)\big\|_p \leq Ct^{-\frac{1}{m}}\big\||x|u_0\big\|_1 ,$$

for all $u_0 \in I\!L^1((1+|x|)dx, I\!R)$.

Now, of this inequality, in view of the density of $I\!L^1((1+|x|)dx, I\!R)$ in $I\!L^1(I\!R)$, and in similar way to [18, 23] it is proven that: if $v_0 \in I\!L^1(I\!R)$ then

$$t^{\frac{1}{m}(1-\frac{1}{p})}\big\|G_m(t) * u_0 - \mathcal{M}G_m(t)\big\|_p \longrightarrow 0\,, \quad \text{when} \quad t \to \infty \tag{7.1}$$

Now, the decay rates from Lemmas 3.5 and 3.6 are extend to the case $p \in [2, \infty]$, with $N = 0$. Indeed, we have

Lemma 7.1 *There exist a constant $C = C(m) > 0$, such that*

$$t^{\frac{1}{m}(1-\frac{1}{p})}\big\|S_\varphi(t) * v_0(x) - G_m(t) * v_0\big\|_p \longrightarrow 0\,, \quad \textit{when} \quad t \to \infty, \tag{7.2}$$

$$t^{\frac{1}{m}(1-\frac{1}{p})}\big\|S(t)v_0(x) - S_\varphi(t) * v_0(x)\big\|_p \longrightarrow 0\,, \quad \textit{when} \quad t \to \infty, \tag{7.3}$$

for all $p \in [2, \infty]$, and $v_0 \in I\!L^1(I\!R) \cap I\!L^2(I\!R)$.

Proof.-For the proof we apply the interpolation inequality

$$\|w\|_p \leq C\|w\|_2^{\frac{2}{p}}\|w\|_\infty^{1-\frac{2}{p}}, \quad \forall\, p \in (2, \infty)\,.$$

When $N = 0$ in the Lemma 3.5, we obtained the case $p = 2$

$$\Big\|S_\varphi(t) * u_0 - G_m(t) * v_0\Big\|_2 \leq Ct^{-1-\frac{1}{2m}}\big\|u_0\big\|_1.$$

Now, we estimate the $I\!L^\infty$-norm. As in (3.17) and (3.21) we decompose $G_m(x,t) = \int_{|\xi|\leq 1} ...d\xi + \int_{|\xi|>1} ...d\xi$, then

$$\Big|S_\varphi(t) * u_0 - G_m(t) * v_0\Big| \leq C\int_{|\xi|\leq 1}\Big|e^{-t|\xi|^m}\Big(e^{\frac{t|\xi|^{2m}}{1+|\xi|^m}} - 1\Big)\Big|\,|\widehat{v_0}|d\xi + \int_{|\xi|>1} e^{-t|\xi|^m}|\widehat{v_0}|d\xi\,.$$

Hence, of the Taylor expansion of the exponential function, $e^x - 1 \leq xe^x$, we have

$$\Big|S_\varphi(t) * u_0 - G_m(t) * v_0\Big| \leq C\int_{|\xi|\leq 1}\Big|e^{-t\frac{|\xi|^m}{1+|xi|^m}}t|\xi|^{2m}\,|\widehat{v_0}|d\xi + \int_{|\xi|>1} e^{-t|\xi|^m}|\widehat{v_0}|d\xi\,.$$

If $|\xi| \le 1$ then $\frac{|\xi|^m}{1+|\xi|^m} \ge \frac{|\xi|^m}{2}$ and if $|\xi| > 1$ then $2|\xi|^m > 1 + |\xi|^m$, hence

$$\Big|S_\varphi(t) * u_0 - G_m(t) * v_0\Big| \le Ct \int_{|\xi|\le 1} e^{-t\frac{|\xi|^m}{2}} |\xi|^{2m} d\xi \|v_0\|_1 + e^{-t} \int_{|\xi|>1} e^{-t\frac{|\xi|^m}{2}} d\xi \|v_0\|_1 \,.$$
$$\le C\Big(t^{-1-\frac{1}{m}} + e^{-t} t^{-\frac{1}{m}}\Big)\|v_0\|_1 \le Ct^{-1-\frac{1}{m}}\|v_0\|_1.$$

Hence, (7.2) is a consequence of this inequality an interpolation formula.
The proff of (7.3), is similar, using Lemma 3.6. □ In consequence, the first term of its asymptotic expansion of the linear solution $S(t)v_0$ is $\mathcal{M}G_m(t)$, of course:

Theorem 7.2 *Let $v_0 \in \mathbb{L}^1(\mathbb{R}) \cap \mathbb{L}^2(\mathbb{R})$. Then the linear solution $v(x,t) = S(t)v_0(x)$ of (2.4) satisfies*

$$t^{\frac{1}{m}(1-\frac{1}{p})}\big\|S(t)v_0(x) - \mathcal{M}G_m(t)\big\|_p \longrightarrow 0\,, \quad \textit{when} \quad t \to \infty$$

for all $p \in [2, \infty]$

Proof.-The relations (7.2) and (7.3) implies that

$$t^{\frac{1}{m}(1-\frac{1}{p})}\big\|S(t)v_0(x) - G_m(t) * v_0(x)\big\|_p \longrightarrow 0\,, \quad \text{when} \quad t \to \infty.$$

This inequality and (7.1) provide the proof of Lemma. □

Lemma 7.3 *Se $a \in (-1, 0]$ e $b \le 0$, existe uma constante $C > 0$, independente de t, tal que*

$$\int_0^t (1+t-\tau)^a (1+\tau)^b d\tau \le \begin{cases} C(1+t)^{a+b+1} & for \quad b > -1; \\ C(1+t)^a & for \quad b < -1\,. \end{cases}$$

Lemma 7.4 *Assume $a \in (-1, 0]$. There exist a constant C independent of t such that*

$$\int_0^t (1+t-\tau)^a (1+\tau)^{-1} d\tau \le C(1+t)^a (1 + log(1+t)).$$

Proof.-After splitting the integral into $\int_0^{1/2} + \int_{1/2}^t$, the above inequality is obtained by estimating each term by the supremum of one of the integrated factors.

The following Theorem say that if $\mathcal{M} = \int u_0(x)dx \ne 0$, then the first term of the asymptotic expansion of the solution $v(x,t)$ of (2.3) is described by the fundamental solution of the linear equation (3.8).

Theorem 7.5 *Let $v = v(x,t)$ be the solution of (2.3) corresponding to the initial data $v_0 \in \mathbb{L}^1(\mathbb{R}) \cap H^2(\mathbb{R})$. Then*

$$t^{\frac{1}{m}(1-\frac{1}{p})}\big\|v(t) - \mathcal{M}G_m(\cdot,t)\big\|_p \le \eta(t) \tag{7.4}$$

$$t^{\frac{1}{m}(q-\frac{1}{p})}\big\|v^q(t) - (\mathcal{M}G_m(\cdot,t))^q\big\|_p \le \eta(t). \tag{7.5}$$

such that $\lim_{t\to\infty} \eta(t) = 0$, with $\mathcal{M} = \int v_0(x)dx$ and $q > m$

Proof.-¿From (6.4)

$$\|v(t)-S(t)v_0\|_p \le \int_0^t \|S_x(t-\tau)K_m\|_p \|v(\tau))\|_\infty^{q-m}\|v(\tau)\|_m^m d\tau. \tag{7.6}$$

¿From (2.6), by interpolation $\|v\|_m \le C(1+t)^{-\frac{1}{m}(1-\frac{1}{m})}$. And by Lemma 7.6 following, we have

$$\|v(t)-S(t)v_0\|_p \le C\int_0^t \big(1+t-\tau\big)^{-\frac{1}{m}(1-\frac{1}{p})-\frac{1}{m}}\big(1+\tau\big)^{-\frac{1}{m}(q-1)}d\tau.$$

Now, from Lemmas 7.4 and 7.3, we have

$$\|v(t)-S(t)v_0\|_p \le C\begin{cases} t^{-\frac{1}{m}(1-\frac{1}{p})+(\frac{m-q}{m})}, \ m<q<m+1 \\ t^{-\frac{1}{m}(1-\frac{1}{p})-\frac{1}{m}}\log t, \ q=m+1 \\ t^{-\frac{1}{m}(1-\frac{1}{p})-\frac{1}{m}}, \ q>m+1 \end{cases}$$

Now, the proof of Theorem 7.5 is an immediate consequence of Theorema 7.2.
The proof of (7.5) result directly from (7.4) and (2.6). Indeed, it suffices to apply a simple consequence of the Lemma 6.5

$$\|v^q(t)-(\mathcal{M}G_m(\cdot,t))^q\|_p \le \|v(t)-\mathcal{M}G_m(\cdot,t)\|_p\Big(\|v(t)\|_\infty^{q-1}+\|\mathcal{M}G_m(\cdot,t)\|_\infty^{q-1}\Big) \qquad \square$$

In the following Lemma, we extend the decay rates from Lemma 6.3 to the case $p\in[0,\infty]$. For this, we use the interpolation inequality

$$\|g\|_p \le C\|g\|_\infty^{1-\frac{1}{p}}\|g\|_1^{\frac{1}{p}} \tag{7.7}$$

When $p=1$, observe that for all smooth rapidly decreasing functions $w=w(x)$ defined in $\mathbb{R}$,

$$\|\widehat{w}\|_1 \le C\|w\|_2^{1/2}\|\partial_x w\|_2^{1/2}. \tag{7.8}$$

The proof of (7.8) can be found e.g. in [8] (example 2).

Lemma 7.6 *Let $m>2$. Then, there exist a positive constant C, independent of t, such that*

$$\|S(t)K_m\|_p \le C(1+t)^{-\frac{1}{m}(1-\frac{1}{p})}, \tag{7.9}$$

$$\|S_x(t)K_m\|_p \le C(1+t)^{-\frac{1}{m}(1-\frac{1}{p})-\frac{1}{m}}, \tag{7.10}$$

for all $p\in[1,\infty]$.

Proof.-For the proof of (7.9) we estimate the $\mathbb{L}^1$-norm of $S(t)K_m$ and then we apply (7.7), for this we use the inequality (7.8) for the function $w(\xi)=\widehat{K_m}e^{-t\Phi(\xi)}$, indeed we have

$$|\partial_\xi w(\xi)| \le C_m\left[t\Big(\frac{|\xi|^{m-1}+|\xi|^m+|\xi|^{2m}}{(1+|\xi|^m)^3}\Big)+\frac{|\xi|^{m-1}}{(1+|\xi|^m)^2}\right]e^{-\frac{t|\xi|^m}{1+|\xi|^m}}.$$

Hence

$$\left|\partial_\xi w(\xi)\right| \leq C_m\big[(t+1)|\xi|^{m-1}\big]e^{-\frac{t|\xi|^m}{2}}, \quad \text{if} \quad |\xi|\leq 1\,,$$
$$\left|\partial_\xi w(\xi)\right| \leq C_m\left[\frac{t|\xi|^{2m}}{\big(1+|\xi|^m\big)^3}+\frac{|\xi|^{m-1}}{\big(1+|\xi|^m\big)^2}\right]e^{-\frac{t}{2}}\,, \quad \text{if} \quad |\xi|>1\,.$$

Then

$$\|\partial_\xi w\|_2^2 = \int_{|\xi|\leq 1}+\int_{|\xi|>1} \leq C(1+t)^2\int_{|\xi|\leq 1}|\xi|^{2(m-1)}]e^{-t|\xi|^m}d\xi$$
$$+Ct^2e^{-t}\int_{|\xi|>1}\frac{|\xi|^{4m}}{\big(1+|\xi|^m\big)^6}d\xi+Ce^{-t}\int_{|\xi|>1}\frac{|\xi|^{2(m-1)}}{\big(1+|\xi|^m\big)^4}d\xi\,.$$

That is

$$\|\partial_\xi w\|_2^2 \leq C(1+t)^{\frac{1}{m}}+Ct^2e^{-t}+Ce^{-t}\leq C(1+t)^{\frac{1}{m}}. \tag{7.11}$$

Now, substituting (6.19) and (7.11) in (7.8), we deduce that

$$\|S(t)K_m\|_1 = \|\widehat{w}\|_1 \leq C\Big((1+t)^{-\frac{1}{2m}}\Big)^{1/2}\Big((1+t)^{\frac{1}{2m}}\Big)^{1/2}\leq C. \tag{7.12}$$

Substituting (7.12) and (6.21) in (7.7) we have

$$\|S(t)K_m\|_p \leq C(1+t)^{-\frac{1}{m}(1-\frac{1}{p})}\,.$$

The proof of (7.10) is similar to (7.9), in this case noting that, if $m>2$

$$\big|S_x(t)K_m\big| \leq C\int\frac{|\xi|\mathrm{e}^{-\frac{t|\xi|^m}{1+|\xi|^m}}}{1+|\xi|^m}d\xi \leq C\int_0^1|\xi|\mathrm{e}^{-\frac{t|\xi|^m}{2}}d\xi+C\int_1^\infty\frac{|\xi|\mathrm{e}^{-\frac{t}{2}}}{1+|\xi|^m}d\xi$$
$$\leq C(1+t)^{-\frac{1}{2m}}+Ce^{-\frac{t}{2}},$$

then

$$\big\|S_x(t)K_m\big\|_\infty \leq C(1+t)^{-\frac{1}{2m}}. \qquad \square$$

Remark 7.7 *When $m=2$, we have that $|\xi|/(1+|\xi|^2)\notin I\!\!L^1(I\!\!R)$, however (7.10) is valid for $p=\infty,\ \ m=2$, (see [21] Lemma 4.2) for more details.*

Lemma 7.8 *Let $G_m(x,t)$ the fundamental solution of the generalized heat equation (3.8). Then, for all $p\in[0,\infty]$,*

$$\big\|S(t)K_m-G_m(t)\big\|_p \leq Ct^{-\frac{1}{m}(1-\frac{1}{p})-\frac{1}{m}}\,, \tag{7.13}$$

$$\big\|\partial_x\big(S(t)K_m-G_m(t)\big)\big\|_p \leq Ct^{-\frac{1}{m}(1-\frac{1}{p})-\frac{2}{m}}\,. \tag{7.14}$$

Proof.-For the proof we use the same argument to that of Lemma 7.6, we omit the details. □

Remark 7.9 *When* $p = 2$*, nothing that (7.13) is a consequence from the Theorem 2.4, indeed, if* $v_0(x) = K_m(x)$*, and as* $\int K_m(x)dx = 1$*, whit* $N = 0$*, we obtain the firs term in the asymptotic expansion i.e.*

$$\left\|S(t)K_m - G_m(t)\right\|_2 \leq t^{-\frac{1}{2m}-\frac{1}{m}} ,$$

Then as consequence of Lemmas 7.6 and 7.8 we have the following Corollary:

Corollary 7.10 *Let* $v(x,t)$ *be the solution to (2.3). Then there exists a constat* $C > 0$ *such that for all* $p \in [0, \infty]$ *and* $q \geq m+1$*,*

$$\left\|S_x(t-\tau)K_m * v^q(\tau)\right\|_p \leq C \begin{cases} (t-\tau)^{-\frac{1}{m}(1-\frac{1}{p})-\frac{1}{m}}\tau^{-\frac{1}{m}(q-1)} ; \\ (t-\tau)^{-\frac{1}{m}(1-\frac{1}{p})}\tau^{-\frac{q}{m}} , \end{cases} \tag{7.15}$$

and

$$\left\|\partial_x\big(S(t-\tau)K_m - G_m(t-\tau)\big) * v^q(\tau)\right\|_p \leq C \begin{cases} (t-\tau)^{-\frac{1}{m}(1-\frac{1}{p})-\frac{2}{m}}\tau^{-\frac{1}{m}(q-1)} ; \\ (t-\tau)^{-\frac{1}{m}(1-\frac{1}{p})-\frac{1}{m}}\tau^{-\frac{q}{m}} , \end{cases} \tag{7.16}$$

for all $t > 0$*,* $\tau \in (0,t)$*.*

Proof.-By interpolation it follows that

$$\left\|v(t)\right\|_p \leq C(1+t)^{-\frac{1}{m}(1-\frac{1}{p})}, \quad \forall\, p \in [2, \infty] .$$

And rememberig that $\left\|v_x(t)\right\|_2 \leq C(1+t)^{-\frac{1}{2m}-\frac{1}{m}}$.
These inequalities combined with Lemmas (7.6) and (7.8) provide the proof of Corollary 7.10.
□

Proof of the Theorem 2.4. We skip the proof of this Theorem because this differs from those Karch [23] in a few technical details, only.

References

[1] L. Abdelouhab, J. L. Bona, M. Felland & J. C. Saut ; *Non local models for nonlinear dispersive waves.* Physica, D40, 360-392 (1989).

[2] Ch. J. Amick, J. L. Bona & M. E. Schonbek; *Decay of solutions of some non linear wave equations.* J. Differential Equations, 81, 1-49 (1989).

[3] T. B. Benjamin; *Lectures on Nonlinear Wave Motions.* in Nonlinear Wave Motions, Alan Newell, ed., Lectures in Applied Mathematics No. 15, AMS. Providence, 3-47 (1974).

[4] T. B. Benjamin; *A new kind of solitary wave.* J. Fluid Mech. 245, 401-411 (1992).

[5] T. B. Benjamin; *Solitary and periodic waves of a new kind.* Philos. Trans. Royal Soc. London A 354, 1775-1806 (1996).

[6] T. B. Benjamin, J. L. Bona & J. J. Mahony; *Model equations for long waves in nonlinear dispersive systems.* Phil. Trans. Roy. Soc. London Ser. A, 272, 47-78 (1972).

[7] P. Biler; *Asymptotic behavior in time of solutions to some equations generalizing Korteweg-de Vries-Burger equations.* Bull. Polish Acad. Sci. Ser. Math., 32, 275-282 (1984).

[8] P. Biler, J. Dziubanski & W. Hebisch *Scattering of small solutions to generalized Benjamin-Bona-Mahony equations in several space dimensions.* Comm. PDE, 17, 1737-1758 (1992)

[9] V. Bisognin & G. Perla Menzala; *Decay rates of the solutions of nonlinear dispersive equations.* Proc. Roy. Soc. Edinburgh Sect. A, 124, 1231-1246 (1994).

[10] J. L. Bona; *Solitary waves and other phenomena associated with model equations for long waves.* Fluid Dynamics Transaction, 10, 77-111 (1980).

[11] J. L. Bona; *On solitary waves and their role in the evolution of long waves.* in Applications of Nonlinear Analysis in the Physical Sciences, H. Amann, N. Bazley, K. Kirchgässner, ed. Pitman Press, London, 183-205 (1981)

[12] J. L. Bona & R. Smith; *The initial-value problem for the Korteweg-de Vries equation.* Phil. Trans. Roy. Soc. London, Ser. A 278, 555-601 (1975).

[13] A. Carpio; *Asymptotic behavior for the vorticity equations in dimensions two and three.* Comm. Partial Differential Equations, 19, 827-872 (1994).

[14] A. Carpio; *Large-time behavior in incompressible Navier-Stokes equations.* SIAM J. Math. Anal., 27, 449-475 (1996).

[15] T. Cazenave & F. B. Weissler; *Asymptotically self-similar global solutions of the nonlinear Schrödinger and heat equations.* Math. Z., 228, No 1, 83-120 (1998).

[16] H. Chen & J. L. Bona; *Existence and asymptotic proprties of solitary-wave solutions of Benjamin-Type equations.* Preprint.

[17] J. Duoandikoetxea & E. Zuazua; *Moments, masses de Dirac et décomposition de fonctions.* C. R. Acad. Sci. Paris, t. 315, Serie I, 693-698 (1992).

[18] M. Escobedo & E. Zuazua; *Large time behavior for convection-diffusion equations in $\mathbb{R}^n$.* J. Funct. Anal., 100, 119-161 (1991).

[19] A. Friedman; *Partial Diferential Equations*. Rinehart and Winston, Inc. New York, 1969.

[20] D. B. Henry; *How to remember the Sobolev inequalities*. Lecture Notes in Mathematics, 957, 97-109, Springer, Berlin, 1982.

[21] G. Karch; *L^p-decay of solutions to dissipative-dispersive perturbations of conservation laws*. Ann. Polon. Math., 67, 65-86 (1997).

[22] G. Karch; *Asymptotic behavior of solutions to some pseudo parabolic equations*. Math. Methods Appl. Sci., 20, 271-289 (1997).

[23] G. Karch; *Large-time behavior of solutions to nonlinear wave equations: higher-order asymptotics*. Math. Methods Appl. Sci., 22, 1671-1697, (1999).

[24] A. Matsumura; *On the asymptotic behavior of solutions of semi-linear wave equation*. Publ. RIMS. Kyoto Univ., 12, 169-189, (1976).

[25] M. Mei; *Asymptotic expansion for Benjamin-Bona-Mahony-Burger equation of generalized type*. Nonlinear Analysis, 33, 699-714, (1998).

[26] M. A. Park; *Large-time behavior of solution for Roseneau-Burger equations*. Nonlinear Analysis, 9, (1995).

[27] R. Prado and E. Zuazua; *Large-time behavior of solution for generalized Benjamin-Bona-Mahony-Burger equations*. Reports LNCC, 9, (2001).

[28] M. E. Schonbek; *Decay of solution to parabolic conservation laws*. Comm. PDE., 7, 449-473 (1980).

[29] M. E. Schonbek; *Uniform decay rates for parabolic conservation laws*. Nonlinear Analysis TMA, 10, 943-953 (1986).

[30] M. E. Schonbek; *Large time behavior of solutions to the Navier-Stokes equations*. Comm. PDE., 11, 733-763 (1986).

[31] I. E. Segal; *Quantization and dispersion for nonlinear relativistic equations*. Mathematical Theory of Elementary Particles, MIT Press, Cambridge, MA. 79-108 (1966)

[32] E. M. Stein; *Singular Integrals and Differentiability Properties of Functions*. Princeton University Press, Princeton: NJ (1970).

[33] T. W. Ting; *Certain non-steady flows of second order fluids*. Arch. Rat. Mech. An., 18, 3-50 (1963).

[34] L. Zhang; *Decay of solutions of generalized Benjamin-Bona-Mahony-Burger equations in n-space dimensions.* Nonlinear Analysis T.M.A., 25, 1343-1396 (1995).

[35] E. Zuazua; *Weakly nonlinear large time behavior in scalar convection-diffusion equations.* Differential Integral Equations, 6, 1481-1491 (1993).

[36] E. Zuazua; *Comportamiento asintótico de ecuaciones escalares de convección-difusión.* Estudos e Comunicações do IMUFRJ 47, Rio de Janeiro, Brasil, 1997.